Some Analytic Aspects Concerning the Collatz Problem
by
Günter Meinardus
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Professor Dr. Günter Meinardus
Fakultät für Mathematik und Informatik
Universität Mannheim
D - 68131 Mannheim
g.meinardus@t-online.de
Abstract. A series of relatively simple equivalences to the Collatz conjecture, concerning the Collatz mapping

$$\tau(n) = \begin{cases} \frac{n}{2} & \text{for even } n, \\ \frac{3n+1}{2} & \text{for odd } n \end{cases}$$

are presented. The conjecture reads as follows: To every $n \in \mathbb{N}$ there exists a number $m \in \mathbb{N}$ such that the $m^{th}$ iterate of $\tau$, applied to $n$, has the value 1. The main topic of this paper consists in investigating a certain linear equation in the space of special Dirichlet series. The conjecture that this equation possesses a null space of dimension 1, generated by the Riemann zeta function, is equivalent to the Collatz conjecture. A number of analytic properties of the operator, which defines the linear equation, is given, some of them concern problems of analytic continuation in the complex domain. A few remarks with respect to generalizations of those problems conclude the paper.

Keywords. The Collatz Problem, the $(3n + 1)$ conjecture, discrete dynamical systems, Dirichlet series, functional equations, analytic number theory.

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1. Introduction. Let $\mathbb{N}$ be the set of natural numbers. We consider the so-called Collatz map $\tau : \mathbb{N} \rightarrow \mathbb{N}$, defined by

$$
\tau(n) = \begin{cases} 
\frac{3n+1}{2} & \text{for even } n \\
\frac{n}{2} & \text{for odd } n 
\end{cases}
$$

(1)

Already in the first half of the last century, it has been conjectured by L. Collatz [3],[4] that to every number $n \in \mathbb{N}$ then exists a number $m \in \mathbb{N}$ such that for the iterates

$$
\tau^{1} = \tau, \quad \tau^{k+1} = \tau \circ \tau^{k}, \quad k \in \mathbb{N}
$$

the assertion

$$
\tau^{m}(n) = 1
$$

(2)

is valid. Until now it seems that this conjecture has neither been proved nor disproved. There is a huge literature on this topic, concerning relations to many fields in mathematics. We here refer to the Lecture Notes in Mathematics, no. 1681 by G. J. Wirsching [10],[11] as a main source.

In this paper we will contribute to such investigations by transforming the problem to a functional equation for special Dirichlet series.

2. Elementary Equivalences. Let $A$ denote the set of all bounded sequences

$$
a = \{a_{\nu}\}_{\nu=1}^{\infty}, \quad a_{\nu} \in \mathbb{R} \quad \text{for } \nu \in \mathbb{N}.
$$

The subset $B \subset A$ is given by such sequences, for which $a_{\nu} \in \{0,1\}$.

If $a = \{a_{\nu}\}_{\nu=1}^{\infty}$ we denote by $a_{\tau}$ the sequence

$$
a_{\tau} = \{a_{\tau(\nu)}\}_{\nu=1}^{\infty}.
$$

Theorem 1. The following assertions are equivalent:

(i) The Collatz conjecture is valid;

(ii) The only solutions of the equation

$$
a = a_{\tau}
$$

(3)

in $A$ are given by the constant sequences

$$
a = \{\lambda\}_{\nu=1}^{\infty} \quad \text{with} \quad \lambda \in \mathbb{R};
$$

(4)
(iii) The equation

\[ a = a_r \]

possesses only the two solutions in \( B \),

\[ a = \{0\}_{\nu=1}^{\infty} \quad \text{and} \quad a = \{1\}_{\nu=1}^{\infty}. \]

**Proof.** We first assume the Collatz conjecture to be true. Then we choose any number \( n \in \mathbb{N} \). According to the assumption there is a number \( m \in \mathbb{N} \) such that \( \tau^m(n) = 1 \). We now consider any solution \( a \) of equation (3). It follows

\[ a_n = a_{r(n)} = \cdots = a_{r^m(n)} = a_1. \]

Since \( n \) was arbitrary chosen, we get \( a_n = a_1 = \lambda = \text{const} \) for all \( n \in \mathbb{N} \). Hence there are no other solutions of equation (3).

Assuming the Collatz conjecture to be false, then there exists a subset \( M \subset \mathbb{N} \), \( M \neq \emptyset \), consisting of all \( \nu \in \mathbb{N} \) for which \( \tau^k(\nu) \neq 1 \) for all \( k \in \mathbb{N} \). We define the sequence

\[ a = \{a_\nu\}_{\nu=1}^{\infty} \]

by

\[ a_\nu = \begin{cases} 1 & \text{for } \nu \in M \\ 0 & \text{for } \nu \notin M \end{cases} \]

Since \( \nu \in M \) implies \( \tau(\nu) \in M \), we get also

\[ a_{r(\nu)} = \begin{cases} 1 & \text{for } \nu \in M \\ 0 & \text{for } \nu \notin M \end{cases} \]

Hence this sequence is a solution of equation (3). Furthermore it is different from all the solutions (4), because of \( 1 \notin M \).

The equivalence of 2.) and 3.) in Theorem 1 is obvious. \( \square \)

Instead of sequences one may consider suitable series, e.g.

\[ \varphi(z) = \sum_{\nu=1}^{\infty} a_\nu \varphi(\nu, z), \]

in some region of the variable \( z \) with some convergence properties. Equivalence theorems concerning the Collatz conjecture can be started by considering functional equations, e.g.

\[ \varphi(z) = \varphi_\tau(z), \]
where
\[ \varphi(z) = \sum_{\nu=1}^{\infty} a_{\nu} \varphi(\alpha z). \]

This has been performed in several papers [1], [2], [7], provided \( \varphi(z) \) stands for a power series around the origin in the complex \( z \)-plane.

We turn over now to the special Dirichlet series
\[ D(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \]
which, for bounded coefficients \( a_n \), converges for all complex variables \( s \) with \( R(s) > 1 \). The vector space of all such Dirichlet series will be denote by \( \hat{A} \). The associated series
\[ D_\tau(s) = \sum_{n=1}^{\infty} a_{\tau(n)} n^{-s} \]
takes, according to (1), the form
\[ D_\tau(s) = \sum_{n=1}^{\infty} a_n (2n)^{-s} + \sum_{\nu=0}^{\infty} a_{3\nu+2} (2\nu + 1)^{-s} = 2^{-s} D(s) + (TD)(s) \]
with the operator \( T : \hat{A} \to \hat{A} \), defined by
\[ (TD)(s) = \sum_{\nu=0}^{\infty} a_{3\nu+2} (2\nu + 1)^{-s}. \]

So we get the

**Theorem 2.** Let \( \hat{B} \) denote the set of Dirichlet series
\[ D(s) = \sum_{n=1}^{\infty} a_n n^{-s} \]
with coefficients \( a_\nu \in \{0, 1\} \) for \( \nu \in \mathbb{N} \), and let \( R(s) > 1 \). Then the following assertions are equivalent:

(i) The Collatz conjecture is valid;

(ii) The equation
\[ D(s) = \frac{1}{1 - 2^{-s}} (TD)(s) \]
possesses in \( \hat{B} \) the only non-trivial solution \( D(s) = \zeta(s) \), where \( \zeta(s) \) denotes the Riemann zeta function.
Proof. We proceed analogously to the proof of Theorem 1.

(i) If the Collatz conjecture is valid, then all the coefficients $a_\nu$ of a non-trivial solution $D \in \hat{D}$ are equal to 1. This shows $D(s) = \zeta(s)$.

(ii) Using the same notations as in the proof of Theorem 1, we see that for the Dirichlet series

$$D(s) = \sum_{\nu \in M} \nu^{-s}$$

we get

$$D_\tau(s) = \sum_{\nu \in M} a_\tau(\nu) \nu^{-s} = \sum_{\nu \in M} \nu^{-s} = D(s).$$

Therefore

$$D(s) = D_\tau(s) = 2^{-s}D(s) + (TD)(s),$$

i.e. the Dirichlet series $D(s)$ yields a solution of equation (6). Since $1 \notin M$ this series is different from $\zeta(s)$ and, according to the assumption $M \neq \emptyset$, not the trivial solution.

This proves Theorem 2. \hfill \Box

In the following section we will give special representations of the operator $T$.

3. A Complex representation of $T$. For real $\gamma$ and a complex function $f : \{ z \in \mathbb{C}, \text{Re}(z) = \gamma \} \rightarrow \mathbb{C}$ we will use the abbreviation

$$\int f(z) \, dz \quad \text{(I)}$$

for the

$$\lim_{\alpha \to \infty} \int_{\gamma - i\alpha}^{\gamma + i\alpha} f(z) \, dz,$$

provided that this limit exists.

Theorem 3. Let $\gamma$ be a real number, $\gamma > 1$. Then, for $D \in \hat{D}$,

$$D(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$
the operator $T$ is represented by the integral

$$
(TD)(s) = \frac{1}{2\pi i} \int_{(\gamma)} D(v) F(v, s) dv, \quad R(s) > \gamma,
$$

with

$$
F(v, s) = \sum_{n=1}^{+\frac{1}{2}} \frac{(3n + 2 + \omega)^{v-1}}{(2n + 1)^{\omega}} d\omega, \quad R(s) > R(v).
$$

**Remark.** The formula (7) is based essentially on the Kronecker-Cahen-Perron Theorem (cp. [5],[9]). We will, however, present a detailed proof. To do this we need some lemmata.

**Lemma 1.** Let $\gamma \in \mathbb{R}$, $\gamma > 0$. Then, for $x \in \mathbb{R}$, $x > 0$, we claim the validity of the formula

$$
\frac{1}{2\pi i} \int_{(\gamma)} \frac{x^v}{v} \, dv = \begin{cases} 
0, & \text{if } 0 < x < 1 \\
\frac{1}{2}, & \text{if } x = 1 \\
1, & \text{if } x > 1
\end{cases}
$$

This is Dirichlet's discontinuous factor.

**Proof of Lemma 1.** For demonstration we draw the figures 1a,b

![Fig.1a](image1)

![Fig.1b](image2)

First we assume $0 < x < 1$. We consider the rectangle with the corners $\gamma - i\alpha$, $\gamma + i\alpha$, $\rho + i\alpha$, $\rho - i\alpha$ (cp fig. 1a), where $\alpha$ and $\rho$ are arbitrary positive numbers and $\rho > \gamma$. Since there are no singularities of the integrand $\frac{x^v}{v}$ in the interior of this rectangle, the value of the corresponding integral equals zero, i.e.

$$
\frac{1}{2\pi i} \int_{W_1} \frac{x^v}{v} \, dv = 0.
$$
Here $W_1$ denotes the circumference of the rectangle. So we have

$$\frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv = \frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\rho - i\alpha} \frac{x^v}{v} dv + \frac{1}{2\pi i} \int_{\rho + i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv + \frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\rho - i\alpha} \frac{x^v}{v} dv + \frac{1}{2\pi i} \int_{\rho + i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv.$$ 

Therefore we may estimate as follows:

$$\left| \frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\rho - i\alpha} \frac{x^v}{v} dv \right| \leq \frac{x^\gamma}{\alpha \pi \log(\frac{1}{x})} + x^\rho \cdot \frac{\pi}{\alpha \rho}.$$ 

For $\rho \to \infty$ we get the inequality

$$\left| \frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv \right| \leq \frac{x^v}{\alpha \pi \log(\frac{1}{x})}$$

for every $\alpha > 0$. It follows for $\alpha \to \infty$,

$$\frac{1}{2\pi i} \int_{\gamma}^{x^v} \frac{v}{v} dv = 0, \quad \text{if } x \in (0,1),$$

i.e. eq.(9a).

Now let $x = 1$. Using the parameter representation

$$v = \gamma + it, \quad -\alpha \leq t \leq \alpha,$$

yields

$$\frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\gamma + i\alpha} \frac{du}{v} = \frac{1}{2\pi} \int_{0}^{\alpha} \left\{ \frac{1}{\gamma + it} + \frac{1}{\gamma - it} \right\} dt = \frac{1}{\pi \gamma} \int_{0}^{\alpha} \frac{dt}{1 + (\frac{t}{\gamma})^2} = \frac{1}{\pi} \int_{0}^{\alpha} \frac{du}{1 + u^2},$$

which tends to $\frac{1}{2}$ for $\alpha \to \infty$. This proves (9b).

Let us consider the third case $x > 1$. This time the rectangle is defined by its corners $\gamma - i\alpha, \gamma + i\alpha, -\rho + i\alpha, -\rho - i\alpha,$ (cp. fig.1b). The pole at $s = 0$ of the integrand belongs to the interior of our rectangle. The residue equals 1. So we get

$$\frac{1}{2\pi i} \int_{W_2} \frac{x^v}{v} dv = 1,$$

where $W_2$ denotes the circumference of the rectangle. Hence

$$\frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv = 1 + \frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\rho - i\alpha} \frac{x^v}{v} dv + \frac{1}{2\pi i} \int_{\rho + i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv + \frac{1}{2\pi i} \int_{\gamma - i\alpha}^{\rho - i\alpha} \frac{x^v}{v} dv + \frac{1}{2\pi i} \int_{\rho + i\alpha}^{\gamma + i\alpha} \frac{x^v}{v} dv.$$
and we gain the estimation
\[
\left| \frac{1}{2\pi i} \int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{x^v}{v} \, dv - 1 \right| \leq \frac{x^\gamma}{\alpha \pi \log(x)} + \frac{\alpha x^{-\rho}}{\pi \rho}.
\]

For \( \rho \to \infty \) we arrive at the inequality
\[
\left| \frac{1}{2\pi i} \int_{\gamma-i\alpha}^{\gamma+i\alpha} \frac{x^v}{v} \, dv - 1 \right| \leq \frac{x^\gamma}{\alpha \pi \log(x)}
\]
for every \( \alpha > 0 \). It follows for \( \alpha \to \infty \):
\[
\frac{1}{2\pi i} \int_{(\gamma)} \frac{x^v}{v} \, dv = 1, \quad \text{if } x > 1,
\]
which is eq. (9c). Thus the Lemma 1 is proved.

\[\square\]

**Remark.** Using the well known formula (cf. [8], eqs. 3.523 resp. 3.527)
\[
I_1(\mu, a) := \int_0^\infty \frac{\cos(\mu t)}{a^2 + t^2} \, dt = \frac{\pi}{2a} e^{-\mu a} \quad \text{for } \mu > 0, a > 0
\]
and
\[
I_2(\mu, a) := \int_0^\infty \frac{\sin(\mu t)}{a^2 + t^2} \, dt = \frac{\pi}{2} e^{-\mu a} \quad \text{for } \mu > 0, a > 0,
\]
the assertions of Lemma 1 follow easily from the identity
\[
\frac{1}{2\pi i} \int_{(\gamma)} \frac{x^v}{v} \, dv = \frac{x^\gamma}{\pi} \left\{ \gamma I_1(\log x, \gamma) + I_2(\log x, \gamma) \right\},
\]
provided \( x > 0 \) and \( x \neq 1 \).

**Lemma 2.** Let the Dirichlet series
\[
D(s) := \sum_{m=1}^{\infty} a_m m^{-s}
\]
be absolutely convergent for all \( s \in \mathbb{C} \) with \( \text{Re}(s) \geq \gamma > 1 \). Then, for every \( x \in \mathbb{R}, x > 0 \) we get the formula
\[
\frac{1}{2\pi i} \int_{(\gamma)} D(v) \frac{x^v}{v} \, dv = \sum_{m \leq x} a_m,
\]
(10)
where
\[ \sum_{m \leq x} a_m := \sum_{m < x} a_m + \begin{cases} \frac{1}{2} a_x & \text{if } x \text{ is an integer} \\ 0 & \text{if } x \text{ is not an integer} \end{cases} \]

**Proof.** According to Lemma 1 we have, for every natural number \( m \),
\[ \frac{1}{2\pi i} \int_{(\gamma)} \left( \frac{x}{m} \right)^v \frac{dv}{v} = \begin{cases} 1 & \text{if } m < x, \\ \frac{1}{2} & \text{if } m = x, \\ 0 & \text{if } m > x, \end{cases} \quad (11) \]

Since \( D(v) \) is uniformly convergent in compact set, contained in the half plane \( R(v) \geq \gamma > 1 \), we get
\[ \frac{1}{2\pi i} \int_{(\gamma)} D(v) \frac{x^v}{v} \, dv = \sum_{m=1}^{\infty} a_m \frac{1}{2\pi i} \int_{(\gamma)} \left( \frac{x}{m} \right)^v \frac{dv}{v} = \sum_{m \leq x} a_m \]

**Proof of Theorem 3.** Let \( m \in \mathbb{N}_0 \). We choose first \( x = 3n + \frac{5}{2} \) and then \( x = 3n + \frac{3}{2} \) and apply Lemma 2. It follows
\[ \frac{1}{2\pi i} \int_{(\gamma)} D(v) \left\{ \left( 3m + \frac{5}{2} \right)^v - \left( 3m + \frac{3}{2} \right)^v \right\} \frac{dv}{v} \]
\[ = \frac{1}{2\pi i} \int_{(\gamma)} D(v) \int_{-\frac{1}{2}}^{+\frac{1}{2}} (3m + 2 + \omega)^{v-1} \, d\omega \, dv = a_{3m+2}. \]

Because of the uniform convergence of the series
\[ \sum_{m=0}^{\infty} \frac{(3m + 2 + \omega)^{v-1}}{(2m + 1)^s} \]
for \( R(s) > R(v) \), we get the following representation of \( T \):
\[ (TD)(s) = \sum_{m=0}^{\infty} \frac{a_{3m+2}}{(2m + 1)^s} = \frac{1}{2\pi i} \int_{(\gamma)} D(v) F(v, s) \, dv, \]
where \( F \) is defined in (8). This proves Theorem 3. \( \Box \)
4. The kernel function $F$. In investigating the properties of the map $T$ one has to study the kernel function $F$. Here we consider again $v$ as a complex parameter and $s$ as the essential complex variable.

The representation of $F$ in eq.(8) by the infinite series shows that $F$ is a holomorphic function with respect to the variable $s$ in the half plane $R(s) > R(v)$. We consider first some analytic continuations of $F$.

**Theorem 4.** The kernel function $F$ with fixed parameter $v$ possesses an analytic continuation with respect to the variable $s$, to a meromorphic function in the half plane $R(s) > R(v) - 1$. The only singularity of the continuation in this region consists in a first order pole at $s = v$ with residue $3^{v-1}/2^v$.

**Proof.** A short consideration shows that it suffices to prove the assertion for the function

$$
\varphi(v, s) := \sum_{m=0}^{\infty} \frac{(3m + 2 + \omega)^{v-1}}{(2m + 1)^s}, \quad R(s) > R(v),
$$

where $\omega \in \mathbb{R}$ belongs to the interval $[-\frac{1}{2}, +\frac{1}{2}]$. Obviously $\varphi$ represents a holomorphic function of $s$ in the half plane $R(s) > R(v)$. Next we consider, for $m \geq 1$, the difference

$$
d_m(v, s) := \frac{(3m + 2 + \omega)^{v-1}}{(2m + 1)^s} - \int_{m-1}^{\infty} \frac{(3t + 2 + \omega)^{v-1}}{(2m + 1)^s} \, dt
$$

$$
= \frac{(3m + 2 + \omega)^{v-1}}{(2m + 1)^s} - \frac{(3m - 1 + 3t + \omega)^{v-1}}{(2m - 1 + 2t)^s} \, dt
$$

$$
= - \int_0^1 t \frac{(3(m + t) - 1 + \omega)^{v-1}}{(2(m + t) - 1)^s+1} \, h(m + t) \, dt
$$

with

$$
h(t) = (6(v - 1) - 6s)t - 3(v - 1) + 2s - 2st \omega.
$$

Here we have used integration by parts. It follows:

(i) The series

$$
\sum_{m=1}^{\infty} d_m(v, s)
$$

converges uniformly in every compact subset of the half plane $R(s) > R(v) - 1$ and hence represents a holomorphic function in that region.
(ii) It is

\[ \varphi(v, s) = \sum_{m=1}^{\infty} d_m(v, s) + (2 + \omega)^{v-1} + \int_0^\infty \frac{(3t + 2 + \omega)^{v-1}}{(2t + 1)^s} \, dt. \]  \quad (12)

We have to investigate the last integral in eq.(12).

One gets

\[
\int_0^\infty \frac{(3t + 2 + \omega)^{v-1}}{(2t + 1)^s} \, dt = \int_0^\infty \frac{(3(1 + t) - 1 + \omega)^{v-1}}{(2(1 + t) - 1)^s} \, dt
\]

\[
= \frac{3^{v-1}}{2^s} \int_0^\infty \frac{(1 + t)^{-s+v-1}}{(1 - \frac{1}{3(1+t)})^s} \, dt
\]

\[
= \frac{3^{v-1}}{2^s} \int_0^\infty \frac{dt}{(1 + t)^{s-v+1}} \, dt + R(v, s)
\]

\[
= \frac{3^{v-1}}{2^s} \frac{1}{(s-v)} + \tilde{R}(v, s).
\]

An expansion of the expression

\[
\frac{(1 - \frac{1-\omega}{3(1+t)})^{v-1}}{(1 - \frac{1}{3(1+t)})^s}
\]

into a power series

\[
\sum_{\nu=0}^{\infty} \frac{c_\nu}{(1+t)^\nu}
\]

leads to the result that \( \tilde{R}(v, s) \) represents a holomorphic function in the region \( R(s) > R(v) - 1 \).

The assertion concerning the first order pole at \( s = v \) is evident. \( \square \)

**Remark.** It may be of interest to mention the recursion formula

\[ \varphi(v, s) = \frac{2}{3} \varphi(v - 1, s - 1) + (\omega + \frac{1}{2}) \varphi(v - 1, s) \]  \quad (13)

for \( R(s) > R(v) \), which can easily be verified.

Using classical tools we investigate the function \( \varphi \), defined in (12), in particular

\[ \varphi(-v, s) = \sum_{m=0}^{\infty} (3m + 2 + \omega)^{-v-1} (2m + 1)^{-s} \]  \quad (14)
in the region
\[ G = \left\{ (v, s) \mid Rv > 0, \ Rs > 0 \right\}. \] (15)

Euler's integral for the \( \Gamma \) function,
\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad Rz > 0, \]
yields
\[ (3m+2+\omega)^{-v-1}(2m+1)^{-s} = \frac{1}{\Gamma(v+1)\Gamma(s)} \int_0^\infty \int_0^\infty x^v y^{s-1} e^{-m(3x+2y)-(2+\omega)x-y} \, dx \, dy. \] (16)

So we get

**Theorem 5.** The representation
\[ \varphi(-v, s) = \frac{1}{\Gamma(v+1)\Gamma(s)} \int_0^\infty \int_0^\infty x^v y^{s-1} e^{-(2+\omega)x-y} \frac{1}{1-e^{-3x-2y}} \, dx \, dy \] (17)
holds true for all
\[(v, s) \in \hat{G},\]
where
\[ \hat{G} = \left\{ (v, s) \in \mathbb{C}^2 \mid Rv > 1, \ Rs > 2 \right\}. \]

**Proof.** We have just to sum in (16) the geometric series
\[ \sum_{m=0}^{\infty} e^{-m(3x+2y)}. \]

The integral representation (17) of \( \varphi(-v, s) \) can be used to gain analytic continuation with respect to \( s \), where \( v \) is fixed and, conversely, with respect to \( v \), where \( s \) is fixed.

We will not go into details here. It may be possible, using classical methods due to Riemann, to get useful estimations of the kernel function \( F(v, s) \), using the representation (17) of \( \varphi(-v, s) \).
5. Generalizations. The mapping (cp. [10], p.13)

\[ \hat{\tau} : \mathbb{N} \to \mathbb{N}, \quad \hat{\tau}(n) = \begin{cases} \frac{n}{2} & \text{for even } n, \\ \frac{3n+1}{2} & \text{for odd } n, \end{cases} \]

for \( n \in \mathbb{N} \) is closely connected with the Collatz mapping \( \tau \), defined in (1). The corresponding kernel function \( \tilde{F}(v, s) \) (cp.eq.(8)) reads

\[ \tilde{F}(v, s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(3n + 1 + \omega)^{v-1}}{(2n + 1)^s} \, d\omega, \quad \text{for } R(s) > R(v). \]

This gives not a big difference to the kernel \( F(v, s) \). On the other hand the full information on the mapping \( \hat{\tau} \) is contained in the kernel \( \tilde{F} \). But: There are three known cycles of \( \hat{\tau} \),

\((1), (5, 7, 10), (17, 25, 37, 55, 82, 41, 61, 91, 136, 68, 34),\)

which gives the reason to conjecture that the dimension of the null space of the equation

\[ D(s) = \frac{1}{1 - 2^{-s}} \cdot \frac{1}{2\pi i} \int_{(\gamma)} D(v) \tilde{F}(v, s) dv \]

is 3.

A slightly more general mapping is given by

\[ \tilde{\tau} : \mathbb{N} \to \mathbb{N}, \quad \tilde{\tau}(n) = \begin{cases} \frac{n}{2} & \text{for even } n, \\ \frac{an + b}{2} & \text{for odd } n, \end{cases} \]

where \( a \in \mathbb{N}, b \in \mathbb{Z} \) and \( a + b \equiv 0 \mod 2 \) and \( a + b > 0 \) holds. Here the corresponding kernel function \( \tilde{F} \) is given by

\[ \tilde{F}(v, s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(an + c + \omega)^{v-1}}{(2n + 1)^s} \, d\omega, \]

where

\[ c = \frac{a + b}{2}. \]

What can be said an the dimension of the null space of the corresponding linear equation for the Dirichlet series? Under which assumptions is this dimension finite?
References