Symmetric Properties in Linear Programming Problems

Werner Oettli \textsuperscript{1} and Maretsugu Yamasaki \textsuperscript{2}

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\textsuperscript{1} Lehrstuhl für Mathematik VII, Universität Mannheim
\textsuperscript{2} Department of Mathematics, Shimane University
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Werner OETTLI and Maretsugu YAMASAKI

Let $X$ and $\tilde{X}$ be real linear spaces which are in duality with respect to a bilinear functional $\langle \cdot , \cdot \rangle$. Likewise let $Y$ and $\tilde{Y}$ be real linear spaces which are in duality with respect to another bilinear functional, for simplicity also denoted by $\langle \cdot , \cdot \rangle$. We assume that the topologies on $X, \tilde{X}$ and $Y, \tilde{Y}$ are such that $X^* = \tilde{X}$, $\tilde{X}^* = X$, $Y^* = \tilde{Y}$, $\tilde{Y}^* = Y$. Let $A : X \to \tilde{Y}$ be a continuous linear mapping. The adjoint $A^* : \tilde{Y} \to \tilde{X}$ is determined by the relation $\langle A^* y, x \rangle := \langle Ax, y \rangle$ for all $x \in X$, $y \in Y$. We require that $A^*$ is continuous and $(A^*)^* = A$. For any nonvoid closed convex cone $\alpha \subseteq X$ we denote by $\alpha^+$ the polar cone of $\alpha$, i.e.,

$$\alpha^+ := \{ \xi \in \tilde{X} | \langle \xi, x \rangle \geq 0 \text{ for all } x \in \alpha \}.$$ 

According to the bipolar theorem, $(\alpha^+)^+ = \alpha$. Furthermore if $\alpha_1 \subseteq \alpha_2$, then $\alpha_2^+ \subseteq \alpha_1^+$, and if $x \in \alpha$, $x \neq 0$, and $\xi \in \text{int} \alpha^+$, then $\langle \xi, x \rangle > 0$. Likewise for any nonvoid closed convex cone $\beta \subseteq Y$ we denote by $\beta^+$ the polar cone of $\beta$, i.e.,

$$\beta^+ := \{ \eta \in \tilde{Y} | \langle \eta, y \rangle \geq 0 \text{ for all } y \in \beta \}.$$ 

The same comments as for $\alpha^+$ apply.

Let $P \subseteq X$ be a fixed nonvoid closed convex cone with $\text{int} P^+ \neq \emptyset$. Let $Q \subseteq Y$ be a fixed nonvoid closed convex cone with $\text{int} Q^+ \neq \emptyset$. Let $P$ be a family of nonvoid closed convex cones $\alpha \subseteq P$ with $\alpha \neq \{0\}$, and let $Q$ be a family of nonvoid closed convex cones $\beta \subseteq Q$ with $\beta \neq \{0\}$. Finally let $f \in \text{int} P^+$ and $g \in \text{int} Q^+$ be given. For $\alpha \in P$, $\beta \in Q$ we consider the following mathematical programming problems:

(1) \[ M(\alpha, \beta) := \sup \{ (f, x) | x \in \alpha, Ax + g \in \beta^+ \}, \]

(2) \[ \bar{M}(\alpha, \beta) := \sup \{ (g, y) | y \in \beta, A^* y + f \in \alpha^+ \}. \]
We remark that problems (1) and (2) are not dual to each other in the usual linear programming sense. Rather, the standard dual of (1) is given by

\[ M^*(\alpha, \beta) := \inf \{ (g, y) | y \in \beta, A^* y + f \in -\alpha^+ \}, \]

and the standard dual of (2) is given by

\[ \tilde{M}^*(\alpha, \beta) := \inf \{ (f, x) | x \in \alpha, Ax + g \in -\beta^+ \}. \]

Let us define

\[ M(P, Q) := \sup \{ M(\alpha, \beta) | \alpha \in P, \beta \in Q \}, \]
\[ \tilde{M}(P, Q) := \sup \{ \tilde{M}(\alpha, \beta) | \alpha \in P, \beta \in Q \}. \]

We shall study the symmetric property \( M(P, Q) = \tilde{M}(P, Q) \).

**Lemma 1.** \( M(\alpha, \beta) > 0 \) and \( \tilde{M}(\alpha, \beta) > 0 \) for all \( \alpha \in P, \beta \in Q \).

Proof: Let \( \tilde{x} \in \alpha, \tilde{x} \neq 0 \). Since \( g \in \text{int } Q^+ \subseteq \text{int } \beta^+ \) we can choose \( \lambda > 0 \) so small that \( \lambda A\tilde{x} + g \in \beta^+ \). Set \( x_0 := \lambda \tilde{x} \). Then \( x_0 \) satisfies the constraints of (1), and from \( x_0 \in \alpha, x_0 \neq 0, f \in \text{int } P^+ \subseteq \text{int } \alpha^+ \) follows \( \langle f, x_0 \rangle > 0 \). Thus \( M(\alpha, \beta) > 0 \). A symmetric argument shows \( \tilde{M}(\alpha, \beta) > 0 \). q.e.d.

We introduce several conditions:

(A.1) For all \( \alpha \in P \), if \( x \in \alpha, x \neq 0, \xi \in -\alpha^+ \), \( \langle \xi, x \rangle = 0 \), then there exists \( \hat{\alpha} \in P \) such that \( \xi \in \hat{\alpha}^+ \).

(A.2) For all \( \beta \in Q \), if \( y \in \beta, y \neq 0, \eta \in -\beta^+ \), \( \langle \eta, y \rangle = 0 \), then there exists \( \hat{\beta} \in Q \) such that \( \eta \in \hat{\beta}^+ \).

Condition (A.1) will be satisfied in particular, if \( P \) contains all cones of the type \( \alpha(\overline{x}) := \{ \lambda \overline{x} | \lambda \geq 0 \} \) with \( \overline{x} \in P, \overline{x} \neq 0 \). Indeed, in this case, if \( x \) and \( \xi \) obey the hypothesis of (A.1), then with \( \hat{\alpha} := \alpha(x) \) we have \( \hat{\alpha} \in P \) and \( \xi \in \hat{\alpha}^+ \), as requested. Likewise condition (A.2) will be satisfied, if \( Q \) contains all cones of the type \( \beta(\overline{y}) := \{ \lambda \overline{y} | \lambda \geq 0 \} \) with \( \overline{y} \in Q, \overline{y} \neq 0 \).

(B.1) For all \( \alpha \in P, \beta \in Q \) the duality theorem holds for (1) and (3), i.e., the linear programming problems (1) and (3) have optimal solutions, and the optimal values \( M(\alpha, \beta) \) and \( M^*(\alpha, \beta) \) are equal.
(B.2) For all \( \alpha \in \mathcal{P}, \beta \in \mathcal{Q} \) the duality theorem holds for (2) and (4), i.e., the linear programming problems (2) and (4) have optimal solutions, and the optimal values \( \tilde{M}(\alpha, \beta) \) and \( \tilde{M}^*(\alpha, \beta) \) are equal.

Conditions (B.1) and (B.2) will be discussed below.

**Theorem 1.** If conditions (A.1), (A.2), (B.1), (B.2) are fulfilled, then the equality \( M(\mathcal{P}, \mathcal{Q}) = \tilde{M}(\mathcal{P}, \mathcal{Q}) \) holds.

Proof: Let \( \alpha \in \mathcal{P}, \beta \in \mathcal{Q} \). Then from (B.1) problem (1) has an optimal solution \( \overline{x} \), problem (3) has an optimal solution \( \overline{y} \), and

\[
(f, \overline{x}) = M(\alpha, \beta) = M^*(\alpha, \beta) = \langle g, \overline{y} \rangle.
\]

From Lemma 1 follows \( \overline{x} \neq 0 \). From the constraints of (1) and (3) follows

\[
\langle f, \overline{x} \rangle \leq -\langle A^* \overline{y}, \overline{x} \rangle = -\langle A \overline{x}, \overline{y} \rangle \leq \langle g, \overline{y} \rangle.
\]

Combined with \( (f, \overline{x}) = \langle g, \overline{y} \rangle \) this gives \( \langle A^* \overline{y} + f, \overline{x} \rangle = 0 \). Since \( \overline{x} \in \alpha, \overline{x} \neq 0 \) and \( A^* \overline{y} + f \in -\alpha^+ \), it follows from (A.1) that \( A^* \overline{y} + f \in \tilde{\alpha}^+ \) for some \( \tilde{\alpha} \in \mathcal{P} \).

From this and \( \overline{y} \in \beta \) follows

\[
M(\alpha, \beta) = \langle g, \overline{y} \rangle \leq \sup \{ \langle g, y \rangle \mid y \in \beta, A^* y + f \in \tilde{\alpha}^+ \}
= \tilde{M}(\tilde{\alpha}, \beta) \leq \tilde{M}(\mathcal{P}, \mathcal{Q}).
\]

Hence \( M(\mathcal{P}, \mathcal{Q}) \leq \tilde{M}(\mathcal{P}, \mathcal{Q}) \). A symmetric argument, using (A.2) and (B.2), gives \( \tilde{M}(\mathcal{P}, \mathcal{Q}) \leq M(\mathcal{P}, \mathcal{Q}) \). Therefore \( M(\mathcal{P}, \mathcal{Q}) = \tilde{M}(\mathcal{P}, \mathcal{Q}) \). q.e.d.

Now we look for a condition which ensures that (B.1) and (B.2) are satisfied simultaneously.

**Lemma 2.** The following conditions are equivalent:

(C.1) \( A^* y \in \text{int}( -P^+) \) for all \( y \in \mathcal{Q}, y \neq 0 \);

(C.2) \( Ax \in \text{int}( -Q^+) \) for all \( x \in \mathcal{P}, x \neq 0 \).
Proof: Because of symmetry it suffices to show that (C.2) implies (C.1).
Let (C.2) hold. Assume, for contradiction, that there exists \( y \in Q, y \neq 0 \) with \( A^*y \notin \text{int} \ (-P^+) \). Then from the separation theorem for convex sets there exists \( x \in (X^*)^* = X, x \neq 0 \), such that
\[
\langle x, A^*y \rangle \geq 0 \geq \langle x, \xi \rangle \quad \text{for all } \xi \in -P^+.
\]
This implies \( \langle Ax, y \rangle \geq 0 \) and \( x \in (P^+)^+ = P \). But from (C.2) follows then \( Ax \in \text{int} \ (-Q^+) \), and therefore \( \langle Ax, y \rangle < 0 \), a contradiction. q.e.d.

**Theorem 2.** Let (C.1) or (C.2) hold. Then both conditions (B.1) and (B.2) are satisfied.

Proof: From Lemma 2 we may assume that both (C.1) and (C.2) are satisfied. Let \( \alpha \in \mathcal{P}, \beta \in \mathcal{Q} \). a) Choose \( x_0 := 0 \). Then \( x_0 \in \alpha \) and \( Ax_0 + g = g \in \text{int} \ Q^+ \).
Now choose \( \tilde{y} \in \beta, \ y \neq 0 \). Then \( \tilde{y} \in Q \), and by (C.1), \( A^*\tilde{y} + U \subseteq \text{int} \ (-P^+) \) for some neighborhood \( U \) of the origin. Choose \( \lambda > 0 \) so large that \( f \in \lambda U \), and set \( y_0 := \lambda \tilde{y} \). Then \( y_0 \in \beta \) and \( A^*y_0 + f \in \lambda A^*\tilde{y} + \lambda U \subseteq \text{int} \ (-P^+) \). Since \( P^+ \subseteq \alpha^+ \) and \( Q^+ \subseteq \beta^+ \) we have altogether obtained \( x_0, y_0 \) such that
\[
x_0 \in \alpha, \quad Ax_0 + g \in \text{int} \ \beta^+,
\]
\[
y_0 \in \beta, \quad A^*y_0 + f \in \text{int} \ (-\alpha^+).
\]
These are the regularity conditions which ensure that the duality theorem holds for (1) and (3) – see [2, p. 164], [3]. Hence (B.1) is satisfied. b) Using (C.2) instead of (C.1) we obtain \( y_0 \) and \( x_0 \) such that
\[
y_0 \in \beta, \quad A^*y_0 + f \in \text{int} \ \alpha^+,
\]
\[
x_0 \in \alpha, \quad Ax_0 + g \in \text{int} \ (-\beta^+).
\]
These are the regularity conditions which ensure that the duality theorem holds for (2) and (4). Hence (B.2) is satisfied. q.e.d.

We turn now to the situation where \( Y = X, \tilde{Y} = \tilde{X} \), so that \( A : X \to \tilde{X} \) and \( A^* : X \to \tilde{X} \). Instead of simply specializing the previous results we consider
a somewhat different problem. From now on let \( \mathcal{P} \) be a family of nonvoid closed convex cones \( \alpha \subseteq X \). Let \( f \in \hat{X}, \ g \in \hat{X} \) be given arbitrarily. For all \( \alpha \in \mathcal{P} \) we consider the problems

\[
(5) \quad L(\alpha) := \sup \{ (f, x) | x \in \alpha, Ax + g \in \alpha^+ \},
\]

\[
(6) \quad \bar{L}(\alpha) := \sup \{ (g, y) | y \in \alpha, A^*y + f \in \alpha^+ \}.
\]

The linear programming dual of (5) is given by

\[
(7) \quad L^*(\alpha) := \inf \{ (g, y) | y \in \alpha, A^*y + f \in -\alpha^+ \},
\]

and the linear programming dual of (6) is given by

\[
(8) \quad \bar{L}^*(\alpha) := \inf \{ (f, x) | x \in \alpha, Ax + g \in -\alpha^+ \}.
\]

We define

\[
(9) \quad L(\mathcal{P}) := \sup \{ L(\alpha) | \alpha \in \mathcal{P} \},
\]

\[
(10) \quad \bar{L}(\mathcal{P}) := \sup \{ \bar{L}(\alpha) | \alpha \in \mathcal{P} \},
\]

and we want to establish the equality \( L(\mathcal{P}) = \bar{L}(\mathcal{P}) \). We require the following conditions:

(D) For all \( \alpha \in \mathcal{P} \) and all \( x \in \alpha \) there exists \( \bar{\alpha} \in \mathcal{P} \) such that \( x \in \bar{\alpha} \subseteq \alpha \) and, whenever \( \xi \in -\bar{\alpha}^+ \) and \( \langle \xi, x \rangle = 0 \), then \( \xi \in \bar{\alpha}^+ \).

(E) For all \( \alpha \in \mathcal{P} \) with \( L(\alpha) > -\infty \) the duality theorem holds for (5) and (7), and for all \( \alpha \in \mathcal{P} \) with \( \bar{L}(\alpha) > -\infty \) the duality theorem holds for (6) and (8).

(F) The suprema occurring in (9) and (10) are finite, and are assumed somewhere on \( \mathcal{P} \).

Now we have:
**Theorem 3.** Let conditions (D), (E), (F) be satisfied. Then $L(P) = \tilde{L}(P)$.

**Proof:** In accordance with condition (F) let $a_1 \in P$ be optimal for $L(P)$, so that $L(P) = L(a_1)$. In accordance with condition (E) let $x$ be optimal for $L(a_1) = \langle f, \tilde{x} \rangle$. Given $x := \tilde{x}$ and $\alpha := a_1$ fix $\tilde{\alpha}$ in accordance with condition (D). Then $x \in \tilde{\alpha} \subseteq \alpha_1$. From the constraints of $L(a_1)$ one has $A\tilde{x} + g \in \alpha_1^+ \subseteq \tilde{\alpha}^+$. Thus $\tilde{x}$ satisfies also the constraints of $L(\tilde{\alpha})$, and therefore $\langle f, \tilde{x} \rangle \leq L(\tilde{\alpha})$. But since $L(P) = \langle f, \tilde{x} \rangle$ it follows that $\langle f, \tilde{x} \rangle = L(\tilde{\alpha})$, and $\tilde{x}$ is also optimal for $L(\tilde{\alpha})$. In accordance with condition (E) let $\tilde{y}$ be optimal for the dual $L^*(\tilde{\alpha})$, so that

\[ \langle f, \tilde{x} \rangle = L(\tilde{\alpha}) = L^*(\tilde{\alpha}) = \langle g, \tilde{y} \rangle. \]

Then $A^* \tilde{y} + f \in -\tilde{\alpha}^+$, and as in the proof of Theorem 1 follows $\langle A^* \tilde{y} + f, \tilde{x} \rangle = 0$. From condition (D) follows $A^* \tilde{y} + f \in \tilde{\alpha}^+$. Consequently $\tilde{y}$ satisfies also the constraints of $L(\tilde{\alpha})$, and therefore $\langle g, \tilde{y} \rangle \leq L(\tilde{\alpha}) \leq L(P)$. Since $L(P) = \langle g, \tilde{y} \rangle$ it follows $L(P) \leq \tilde{L}(P)$. A symmetric argument gives $\tilde{L}(P) \leq L(P)$. Hence the claimed equality is true. q.e.d.

Let us discuss condition (D). It is satisfied for instance, if $\alpha \subseteq P$ for all $\alpha \in P$ and $P$ contains all cones of the type $\alpha(\tilde{x}) := \{ \lambda \tilde{x} | \lambda \geq 0 \}$, $\tilde{x} \in P$, where $P \subseteq X$ is a given nonvoid closed convex cone. Indeed, if $x \in \alpha$ for some $\alpha \in P$, then choosing $\tilde{\alpha} := \alpha(x)$ one has $\tilde{\alpha} \in P$, $x \in \tilde{\alpha} \subseteq \alpha$, and if $\langle \xi, x \rangle = 0$, then $\xi \in \tilde{\alpha}^+$ ($x = 0$ is permitted here since $\alpha = \{ \epsilon \}$ is not excluded). Hence (D) is satisfied.

Another situation where (D) is satisfied is the following. Let $K$ be a finite set and $X := M^K$, $P$ be the family of all cones of the type $\alpha(A) := \{ z \in M^K | x_i \geq 0 \}$ for all $i \not\in A$, $x_i = 0$ for all $i \in K \setminus A \}$, where $A$ runs over all subsets of $K$. Then $(\alpha(A))^+ = \{ y \in M^K | y_i \geq 0 \}$ for all $i \in A$. For $x \in M^K$ let $\text{supp } x := \{ i \in K | x_i > 0 \}$. Now if $x \in \alpha(A)$, then choosing $\tilde{\alpha} := \alpha(\text{supp } x)$ we have $\tilde{\alpha} \in P$ and $x \in \tilde{\alpha} \subseteq \alpha(A)$. Moreover, if $\xi \in -\tilde{\alpha}^+$ and $\langle \xi, x \rangle = 0$, then $\xi_i = 0$ for all $i \in \text{supp } x$, hence $\xi \in \tilde{\alpha}^+$: (D) is satisfied. In this situation the
conclusion of Theorem 3 is equivalent with
\[ \sup \{ \langle f, x \rangle | x \in \mathbb{R}_+^K, (Ax + y)_i \geq 0 \text{ for all } i \in \text{supp } x \} \]
\[ = \sup \{ \langle g, y \rangle | y \in \mathbb{R}_+^K, (A^*y + f)_j \geq 0 \text{ for all } j \in \text{supp } y \}, \]
provided that both suprema are finite and are assumed. An infinite-dimensional analog of this result with \( K \) a compact Hausdorff space and \( x, y \) Radon measures over \( K \), has been given by Ohtsuka [4], and motivated the present investigation.

References


Lehrstuhl für Mathematik VII
Universität Mannheim
Mannheim
Germany

and

Department of Mathematics
Shimane University
Matsue
Japan