ON THE DYNAMICS
OF CONTINUOUS DISTRIBUTIONS
OF DISLOCATIONS

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On the Dynamics of Continuous Distributions of Dislocations

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Abstract

For materials with a continuous distribution of dislocations, equations of motion are derived from a symplectic structure on an appropriate configuration space. The proposed dynamics generalizes from elasticity.

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1 Introduction

A mathematical framework for the dynamics of an elastic material is given by the space of all embeddings $E(M; \mathbb{R}^3)$ of a reference body $M$ into the physical space $\mathbb{R}^3$. The constitutive law determining the equations of evolution can be given in terms of a virtual work functional on this phase space, cf. [8]. The invariance of the system under rigid global translations implies that the differential $dj$ of the embedding $j \in E(M; \mathbb{R}^3)$ is the essential quantity for the constitutive behaviour of the material, cf. [3]. In classical terms this differential is precisely the deformation gradient of the actual configuration of the system. Mathematically the deformation gradient $dj$ may be considered as an exact ($\mathbb{R}^3$-valued) differential one-form in $\Omega^1(M; \mathbb{R}^3)$.

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In the continuum theory of defects one describes dislocations by a torsion density on the reference body, cf. [9, 15, 19]. This torsion density may be identified with an exact (\( \mathbb{R}^3 \)-valued) differential two-form \( d\gamma \in \Omega^2(M; \mathbb{R}^3) \). The corresponding Burgers vector computes as the integral of \( d\gamma \) over a bounded surface \( S \subset M \), cf. [20].

To incorporate this description of dislocations into the framework of elasticity, the Helmholtz decomposition theorem is utilized which claims that any differential form may be uniquely decomposed into a gradient and a divergence-free part. A generalised configuration space for a material with dislocations \( \mathcal{V}(M; \mathbb{R}^3) \) is defined as a submanifold of \( \Omega^1(M; \mathbb{R}^3) \). Each generalised configuration \( \gamma \in \mathcal{V}(M; \mathbb{R}^3) \) splits into an elastic or gradient part \( dj \), where \( j \in E(M; \mathbb{R}^3) \) is an embedding, and into a so-called plastic part \( \beta \) describing the dislocation density, cf. [20].

The main objective of this paper is to derive a dynamics for a material with a continuous distribution of dislocations. This is done by introducing a symplectic structure \( \Omega \) and a kinetic energy functional \( \mathcal{E} \) on the tangent space \( T\mathcal{V}(M; \mathbb{R}^3) \) of the configuration space \( \mathcal{V}(M; \mathbb{R}^3) \). The constitutive behaviour of such a system is described by a virtual work functional \( F \) on \( \mathcal{V}(M; \mathbb{R}^3) \). The resulting principle of virtual work determines weak equations of motion for the generalised configurations \( \gamma \).

Using the Helmholtz decomposition theorem, these equations split into a part which determines the evolution of the elastic parts \( dj \) of a generalised configuration \( \gamma \) and into a part which determines the evolution of the plastic parts \( \beta \). The equations for the elastic parts are just the well-known equations in classical elasticity. Thus, for purely elastic materials, this approach covers the classical theory.

2 Differential Forms

Since in this approach towards a dynamics of dislocations, differential forms provide a convenient framework, a brief introduction is given. Let \( M \) be the body manifold in the sense of elasticity. Assume that \( M \) is a smooth connected 3-dimensional compact oriented Riemannian manifold with boundary which is embedable into the physical space \( \mathbb{R}^3 \). A \( \mathbb{R}^3 \)-valued differential form \( \omega \in \Omega^k(M; \mathbb{R}^3) \) of degree \( k \) is a smooth assignment of a skew-symmetric \( k \)-linear map \( \omega_p \) on \( T_p M \) to each point \( p \in M \), where

\[
\omega_p : T_p M \times \cdots T_p M \longrightarrow \mathbb{R}^3 \quad \forall p \in M.
\]

In classical terms, differential forms may be considered as skew-symmetric two-point tensors of type \((1, k)\) on the body manifold \( M \) which are well-known objects in continuum mechanics, cf. [12]. Of particular interest in our approach are the cases \( k = 0, 1, 2 \). For example, the deformation gradient and the first Piola-Kirchoff stress tensor are
considered here as $\mathbb{R}^3$-valued one-forms on the body manifold $M$, i.e. as some $\omega \in \Omega^1(M; \mathbb{R}^3)$. Analogously, placements of $M$ and force fields are elements in $\Omega^0(M; \mathbb{R}^3)$ which, by definition, is equal to $C^\infty(M; \mathbb{R}^3)$.

Each $\Omega^k(M; \mathbb{R}^3)$ may be equipped with a fibre metric by using the Riemannian metric $g$ on $M$ and the standard scalar product $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ on $\mathbb{R}^3$. For our purposes, it suffices to consider the cases $k = 0, 1$. Let $E_1, E_2, E_3 \in \Gamma(TM)$ be a triple of vector fields orthonormal with respect to the metric $g$. A fibre metric on $\Omega^1(M; \mathbb{R}^3)$ is then defined by

$$\langle \omega, \eta \rangle := \sum_i \langle \omega(E_i), \eta(E_i) \rangle_{\mathbb{R}^3}, \quad \omega, \eta \in \Omega^1(M; \mathbb{R}^3).$$

(1)

The product (1) does only depend on the metric $g$ but not on the chosen frame on $M$, cf. [13]. Notice that (1) corresponds to the contraction of skew symmetric two-point tensors. If $e_1, e_2, e_3 \in \mathbb{R}^3$ denotes the standard basis in $\mathbb{R}^3$ and $\theta^1, \theta^2, \theta^3 \in \Omega^1(M)$ the dual frame corresponding to $E_1, E_2, E_3$, then, in coordinates, any one-forms $\omega$ and $\eta$ may be written as $\omega = \sum_{L, i} \omega^i_L e_L^i$ and $\eta = \sum_{L, i} \eta^i_L e_L^i$. Thus (1) reads

$$\langle \omega, \eta \rangle = \sum_{L, i=1}^3 \omega^i_L \eta^i_L.$$

With the help of the Riemannian volume element $\mu$ induced by $g$, the space $\Omega^1(M; \mathbb{R}^3)$ is now endowed with an $L^2$-product $\mathcal{G}$, given by

$$\mathcal{G}(\omega, \eta) := \int_M \langle \omega, \eta \rangle \mu, \quad \omega, \eta \in \Omega^1(M; \mathbb{R}^3).$$

(2)

For $k = 0$ the corresponding $L^2$-product $\mathcal{G}$ is just the usual one. Let $\nabla$ denote the Levi-Civita connection on $M$ associated to $g$. Then $\nabla$ induces a covariant derivative on $\Omega^1(M; \mathbb{R}^3)$, given by

$$\nabla_Y \omega(X) = D[\omega(X)](Y) - \omega(\nabla_Y X), \quad X, Y \in \Gamma(TM).$$

Here, the first term of the right hand side means the directional derivative of the $\mathbb{R}^3$-valued function $\omega(X)$ in direction of the vector field $Y$. For $k = 0$ the second term of the right hand side of the above expression vanishes. The covariant derivative allows to write the exterior derivative $d : \Omega^1(M; \mathbb{R}^3) \rightarrow \Omega^2(M; \mathbb{R}^3)$ as

$$d\omega(X, Y) = (\nabla_X \omega)(Y) - (\nabla_Y \omega)(X), \quad X, Y \in \Gamma(TM).$$

For $k = 0$ the exterior derivative corresponds to the gradient. The co-differential $\delta : \Omega^1(M; \mathbb{R}^3) \rightarrow \Omega^0(M; \mathbb{R}^3)$ may be defined by

$$\delta \omega := - \sum_{i=1}^3 (\nabla_{E_i} \omega)(E_i).$$
Notice that the co-differential $\delta$, unlike the exterior derivative, depends on the chosen Riemannian metric $g$. In classical tensor notation, $\delta$ corresponds to the divergence of a tensor field.

Let $\mathcal{N}$ denote the outward pointing unit normal field on the boundary $\partial M$ of $M$. A differential one-form $\omega$ is called parallel to $\partial M$ iff its normal component vanishes, that is $\omega(\mathcal{N}) = 0$. Define the space of all divergence-free and parallel one-forms by

$$ \mathcal{D}(M; \mathbb{R}^3) := \{ \omega \in \Omega^1(M; \mathbb{R}^3) \mid \delta \omega = 0 \text{ and } \omega(\mathcal{N}) = 0 \}. $$

We are now able to state the Helmholtz decomposition for the special case of $\mathbb{R}^3$-valued one-forms. For a general proof see [17].

**Theorem 2.1** *Helmholtz Decomposition*

Let $M$ be a compact, oriented Riemannian manifold with boundary. Then for any $\omega \in \Omega^1(M; \mathbb{R}^3)$ there exist $\theta \in \Omega^0(M; \mathbb{R}^3)$ and $\beta \in \mathcal{D}(M; \mathbb{R}^3)$ such that $\omega = d\theta + \beta$. Moreover, $d\theta$ and $\beta$ are mutually $L^2$-orthogonal with respect to the inner product (2), that is the decomposition

$$ \Omega^1(M; \mathbb{R}^3) = d\Omega^0(M; \mathbb{R}^3) \oplus \mathcal{D}(M; \mathbb{R}^3) $$

is direct and $L^2$-orthogonal.

### 3 The Kinematics of Dislocations

Let $j : M \rightarrow \mathbb{R}^3$ be a smooth embedding of the body manifold $M$ into the Euclidean space $\mathbb{R}^3$, and $E(M; \mathbb{R}^3)$ denote the space of all such embeddings. In pure elasticity $E(M; \mathbb{R}^3)$ constitutes the configuration space of the system; in classical terms its elements $j$ are called placement (or transplacement) fields. The displacement fields $u \in C^\infty(M; \mathbb{R}^3)$ compute as $u = (j - j_0)$, where $j_0$ is a reference configuration.

This section is aimed at generalising the classical configuration space $E(M; \mathbb{R}^3)$ in such a way that the description of the kinematics of dislocations is included. We introduce a configuration space for an elastic solid whose internal structure is characterised by a frame, i.e. a triple of linear independent vector fields on $M$

$$ Y_1, Y_2, Y_3 \in \Gamma(TM). \quad (3) $$

Physically, these vector fields describe lattice vectors of a continuised crystal as worked out in [9]. We denote the standard basis of $\mathbb{R}^3$ by $e_1, e_2, e_3$. Since $M$ is embedable

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1. $E(M; \mathbb{R}^3)$ is an open subset in the Fréchet space $C^\infty(M; \mathbb{R}^3)$, see [2] for details.
into $\mathbb{R}^3$, for any arbitrary frame (3), there exists a unique fibrewise one-to-one map
\[ \gamma : TM \rightarrow \mathbb{R}^3 \]
such that
\[ \gamma_p(Y_i(p)) = e_i, \quad i = 1, 2, 3 \quad \forall p \in M. \] (4)
Mathematically, $\gamma$ is a $\mathbb{R}^3$-valued one-form $\gamma \in \Omega^1(M; \mathbb{R}^3)$ on $M$ which is fibrewise one-to-one. The set of all these one-forms is defined by
\[ \mathcal{I}(M; \mathbb{R}^3) := \{ \gamma \in \Omega^1(M; \mathbb{R}^3) \mid \gamma_p : T_pM \rightarrow \mathbb{R}^3 \text{ is one-to-one, } p \in M \}. \]
Consider a fixed $\gamma \in \mathcal{I}(M; \mathbb{R}^3)$. Then $\gamma(X) \in C^\infty(M; \mathbb{R}^3)$ is a smooth function for each $X \in \Gamma(TM)$. Let $D(\gamma(X))(Y)$ denote the directional derivative of $\gamma(X)$ into the direction of some $Y \in \Gamma(TM)$. A connection $\nabla[\gamma]$ on $TM$ associated with $\gamma \in \mathcal{I}(M; \mathbb{R}^3)$ is then defined by
\[ \nabla[\gamma]_Y X = \gamma^{-1} D(\gamma(X))(Y), \quad X, Y \in \Gamma(TM). \] (5)
In a coordinate system on $M$, the Christoffel symbols of (5) read
\[ \Gamma^k_{lm} = \sum_{\lambda=1}^3 (\gamma^{-1})^k_{\lambda} \partial_l \gamma^\lambda_m. \]
It is easy to verify that the curvature of this connection vanishes, i.e. the connection (5) is flat. Conversely, it is shown in [20] that for any flat connection $\nabla$ on $TM$, there is some $\gamma \in \mathcal{I}(M; \mathbb{R}^3)$ with $\nabla = \nabla[\gamma]$. The torsion $T^\nabla$ of an arbitrary connection $\nabla$ is defined by
\[ T^\nabla(X, Y) = \nabla_Y X - \nabla_X Y - [X, Y] \quad \forall X, Y \in \Gamma(TM). \]
In particular, if $T[\gamma]$ denotes the torsion of $\nabla[\gamma]$, it follows from (5) and the definition of the exterior derivative $d$ that
\[ d\gamma(X, Y) = \gamma(T[\gamma](X, Y)), \quad X, Y \in \Gamma(TM). \]
In classical terms, the torsion of a connection describes the dislocation density or the material inhomogeneity of a material. Since $\gamma$ is fibrewise one-to-one, the discussion shows that $T[\gamma] = \gamma^{-1} d\gamma$. Therefore, the dislocation density $T[\gamma]$ might as well be measured by the exterior derivative of the one-form $\gamma \in \mathcal{I}(M; \mathbb{R}^3)$. Hence, the two-form $d\gamma$ will be referred to as a dislocation density of the material. In particular,
\[ d\gamma = 0 \quad \iff \quad T[\gamma] = 0, \]
implying that the material is defect-free if and only if $\gamma$ is closed, i.e. $d\gamma = 0$. The Burgers vector $b$ of an arbitrary surface $S \subset M$ associated with the dislocation density $d\gamma$ computes as the integral
\[ b = \int_S d\gamma. \]
The crucial observation is that according to the Helmholtz decomposition, Theorem 2.1, each \( \gamma \in \mathcal{I}(M; \mathbb{R}^3) \) uniquely splits into
\[
\gamma = dv + \beta, \quad \text{where} \quad dv \in d\Omega^0(M; \mathbb{R}^3), \quad \beta \in \mathcal{D}(M; \mathbb{R}^3).
\] (6)
Since \( d^2 = 0 \), only the divergence-free part \( \beta \in \mathcal{D}(M; \mathbb{R}^3) \) of \( \gamma \) contributes to the dislocation density. In particular \( d\gamma = d\beta \), i.e. the dislocation density is uniquely determined by the so-called non-exact component \( \beta \).

As far as classical elasticity is concerned, the essential quantity for the constitutive behaviour of a material is the deformation gradient \( dj \in \Omega^1(M; \mathbb{R}^3) \) of an actual embedding \( j \in E(M; \mathbb{R}^3) \). It is shown in [3] that the set of all such gradients
\[
dE(M; \mathbb{R}^3) = \{ dj \mid j \in E(M; \mathbb{R}^3) \}
\]
is an open subset of the Fréchet space of all one-forms \( \Omega^1(M; \mathbb{R}^3) \). Since differentials of embeddings are fibrewise one-to-one, we have \( dE(M; \mathbb{R}^3) \subseteq I(M; \mathbb{R}^3) \). Each deformation gradient \( dj \in dE(M; \mathbb{R}^3) \) defines a frame \( X_1, X_2, X_3 \in \Gamma(TM) \) by solving
\[
dj(X_l) = e_l, \quad l = 1, 2, 3.
\] (7)
Since \( d^2 = 0 \), it follows from (4) that this triple of vector fields characterises a defect-free material. Therefore, a placement \( j \in E(M; \mathbb{R}^3) \) will be called integrable configuration of the body manifold \( M \); an arbitrary \( \gamma \in \mathcal{I}(M; \mathbb{R}^3) \) will be referred to as a generalised configuration of \( M \).

According to [18] the evolution of defects is held responsible for the discrepancy between the macroscopic deformation and the behaviour of the lattice. Therefore, we think of the component \( \beta \in \mathcal{D}(M; \mathbb{R}^3) \) as a quantity by which the frame \( X_1, X_2, X_3 \) is incompatibly deformed. The vector fields
\[
(dj + \beta)(X_1), (dj + \beta)(X_2), (dj + \beta)(X_3)
\]
constitute a frame on \( j(M) \subset \mathbb{R}^3 \) if and only if \( dj + \beta \) is injective. For \( \beta \neq 0 \), this frame represents a dislocated lattice on the embedded body.

The general idea is that only the integrable part, i.e. the gradient part of a generalised configuration \( \gamma \in \mathcal{I}(M; \mathbb{R}^3) \) becomes visible as a placement of the body manifold in Euclidean space. Thus, we consider generalised configurations \( \gamma = dj + \beta \in \mathcal{I}(M; \mathbb{R}^3) \) whose integrable part \( dj \) stems from a placement \( j \in E(M; \mathbb{R}^3) \) and whose non-integrable part \( \beta \) lies in \( \mathcal{D}(M; \mathbb{R}^3) \). The set of all such configurations is denoted by
\[
\mathcal{V}(M; \mathbb{R}^3) = \{ dj + \beta \in \mathcal{I}(M; \mathbb{R}^3) \mid j \in E(M; \mathbb{R}^3), \beta \in \mathcal{D}(M; \mathbb{R}^3) \}.
\]
Observe that by construction \( \mathcal{V}(M; \mathbb{R}^3) \subset \mathcal{I}(M; \mathbb{R}^3) \), where the exact parts of generalised configurations \( \gamma \in \mathcal{V}(M; \mathbb{R}^3) \) are restricted to embeddings \( j \in E(M; \mathbb{R}^3) \). Since \( \mathcal{V}(M; \mathbb{R}^3) \) is an open Fréchet submanifold of \( \Omega^1(M; \mathbb{R}^3) \), we take \( \mathcal{V}(M; \mathbb{R}^3) \) as a configuration space for an elastic material which possibly may be dislocated, cf. [20].
4 The Geometry of $\mathcal{V}(M; \mathbb{R}^3)$

For a mathematical formulation of a dynamic theory of dislocated materials, a metric on the configuration space $\mathcal{V}(M; \mathbb{R}^3)$ is needed. Following [6], we first introduce an appropriate metric on $dE(M; \mathbb{R}^3)$. Let $\rho : M \rightarrow \mathbb{R}$ be a strictly positive real-valued function which physically may be thought of as the mass distribution of the material. Since $E(M; \mathbb{R}^3)$ is open in $C^\infty(M; \mathbb{R}^3)$, the tangent manifold of $E(M; \mathbb{R}^3)$ is trivial

$$TE(M; \mathbb{R}^3) = E(M; \mathbb{R}^3) \times C^\infty(M; \mathbb{R}^3).$$

Identifying each tangent vector with its principal part, a metric on $E(M; \mathbb{R}^3)$ is defined by setting

$$G_\rho(u_1, u_2) := \int_M \rho(u_1, u_2) \\mu, \quad u_1, u_2 \in C^\infty(M; \mathbb{R}^3). \quad (8)$$

Using (8), each $j \in E(M; \mathbb{R}^3)$ and each $u \in C^\infty(M; \mathbb{R}^3)$ may be decomposed into

$$j = j^0 + C_j, \quad \text{where} \quad C_j \in \mathbb{R}^3, \quad G_\rho(j^0, c) = 0 \quad \forall c \in \mathbb{R}^3$$

and

$$u = u^0 + C_u, \quad \text{where} \quad C_u \in \mathbb{R}^3, \quad G_\rho(u^0, c) = 0 \quad \forall c \in \mathbb{R}^3$$

respectively. The sets

$$E_0(M; \mathbb{R}^3) := \{ j \in E(M; \mathbb{R}^3) \mid \int_M \rho j \mu = 0 \}$$

and

$$C_0^\infty(M; \mathbb{R}^3) := \{ u \in C^\infty(M; \mathbb{R}^3) \mid \int_M \rho u \mu = 0 \}$$

are Fréchet manifolds which are naturally isomorphic to $dE(M; \mathbb{R}^3)$ and $d\Omega^0(M; \mathbb{R}^3)$ respectively, cf. [3, 4]. Since $dE(M; \mathbb{R}^3) \subset d\Omega^0(M; \mathbb{R}^3)$ is open,

$$T(dE(M; \mathbb{R}^3)) = dE(M; \mathbb{R}^3) \times d\Omega^0(M; \mathbb{R}^3).$$

Configurations in $j \in E_0(M; \mathbb{R}^3)$ are such that the center of mass is kept fixed, $C_j = 0$. A metric on $dE(M; \mathbb{R}^3)$ naturally induced by this construction is given by

$$G_E(du_1, du_2) := \int_M \rho(u_1, u_2) \\mu, \quad du_1, du_2 \in d\Omega^0(M; \mathbb{R}^3), \quad (9)$$

where we identify tangent vectors with their principal parts.

As the configuration space $\mathcal{V}(M; \mathbb{R}^3)$ is an open subset of $\Omega^1(M; \mathbb{R}^3)$, the tangent manifold $T\mathcal{V}(M; \mathbb{R}^3)$ of $\mathcal{V}(M; \mathbb{R}^3)$ is trivial

$$T\mathcal{V}(M; \mathbb{R}^3) = \mathcal{V}(M; \mathbb{R}^3) \times \Omega^1(M; \mathbb{R}^3).$$

Applying Theorem 2.1, tangent vectors $\eta \in T\mathcal{V}(M; \mathbb{R}^3)$ allows to equip the configuration space $\mathcal{V}(M; \mathbb{R}^3)$ with a metric as follows.
Definition 4.1 Let $\gamma \in \mathcal{V}(M; \mathbb{R}^3)$ be an arbitrary generalised configuration. For each pair $\eta_i \in T_i \mathcal{V}(M; \mathbb{R}^3)$, $i = 1, 2$, let

$$\eta_i = du_i + v_i$$

with $du_i \in d\Omega^0(M; \mathbb{R}^3)$, $v_i \in \mathcal{D}(M; \mathbb{R}^3)$ be the respective Helmholtz decompositions. A metric $\mathcal{G}_\gamma$ on the configuration space $\mathcal{V}(M; \mathbb{R}^3)$ is defined by setting

$$\mathcal{G}_\gamma[\gamma](\eta_1, \eta_2) := \mathcal{G}_\gamma^{(e)}[\gamma](du_1, du_2) + \mathcal{G}_\gamma^{(p)}[\gamma](v_1, v_2).$$

The elastic part of $\mathcal{G}^{(e)}$ is given by

$$\mathcal{G}^{(e)}_\gamma[\gamma](du_1, du_2) := \mathcal{G}_E(du_1, du_2),$$

where $\mathcal{G}_E$ is defined in (9). The plastic part of $\mathcal{G}_\gamma$ is given by

$$\mathcal{G}^{(p)}_\gamma[\gamma](v_1, v_2) := \int_M \sigma(v_1, v_2)\mu, \quad v_1, v_2 \in \mathcal{D}(M; \mathbb{R}^3),$$

where $\sigma \in C^\infty(M)$ is a strictly positive real-valued function.

Notice that physically, the function $\sigma$ appearing in the above metric may be thought of as the density of inertia of the dislocations. For sake of simplicity we assume that the density $\sigma$ is independent of the actual configuration. This means that all dislocations respond to a force action by the same specific inertia.

Let $T \tau_{\gamma} : T^2 \mathcal{V}(M; \mathbb{R}^3) \rightarrow T \mathcal{V}(M; \mathbb{R}^3)$ denote the tangent map of the canonical projection $\tau_{\gamma}$ and $\tau_{\mathcal{V}} \mathcal{V}(M; \mathbb{R}^3)) := \ker T \tau_{\gamma}$ the vertical bundle. Moreover, let $V \mathcal{X} \subset V(T \mathcal{V}(M; \mathbb{R}^3))$ denote the vertical component of any vector $\mathcal{X} \in T^2 \mathcal{V}(M; \mathbb{R}^3)$. The metric $\mathcal{G}_\gamma$ given in Definition 4.1 defines a natural weakly nondegenerate symplectic two-form $\Omega$ on $T \mathcal{V}(M; \mathbb{R}^3)$ by

$$\Omega[\xi](\mathcal{X}, \mathcal{Y}) := \mathcal{G}_\gamma[\gamma](V \mathcal{X}, T \tau_{\gamma} \mathcal{X}) - \mathcal{G}_\gamma[\gamma](V \mathcal{Y}, T \tau_{\gamma} \mathcal{Y})$$

for all $\mathcal{X}, \mathcal{Y} \in T_i T \mathcal{V}(M; \mathbb{R}^3)$, $\xi \in T_i \mathcal{V}(M; \mathbb{R}^3)$, $\gamma \in \mathcal{V}(M; \mathbb{R}^3)$. Thus, $T \mathcal{V}(M; \mathbb{R}^3)$ endowed with $\Omega$ becomes a symplectic manifold. Since $T \mathcal{V}(M; \mathbb{R}^3)$ is trivial, in coordinates one has

$$\mathcal{X} = (\gamma, \xi, \xi_1, \xi_2)$$

and

$$\mathcal{Y} = (\gamma, \xi, \eta_1, \eta_2)$$

which in turn yields

$$\Omega[\gamma, \xi](((\xi_1, \xi_2), (\eta_1, \eta_2)) = \mathcal{G}_\gamma[\gamma](\eta_2, \xi_1) - \mathcal{G}_\gamma[\gamma](\xi_2, \eta_1).$$

8
The metric $g$ induces the kinetic energy functional $E : T\mathcal{V}(M; \mathbb{R}^3) \rightarrow \mathbb{R}$ of the dislocated material by setting

$$E(\xi) := \frac{1}{2} g[\gamma](\nu, \nu), \quad \xi \in T_\gamma \mathcal{V}(M; \mathbb{R}^3), \quad \gamma \in \mathcal{V}(M; \mathbb{R}^3).$$

(11)

If $\xi = du + \nu$ denotes the Helmholtz decomposition, then, according to Definition 4.1, the kinetic energy $E$ of a dislocated material splits into an elastic part

$$E^{(e)}(\xi) := \frac{1}{2} g^{(c)}[\gamma](du, du),$$

and into a plastic part

$$E^{(p)}(\xi) := \frac{1}{2} g^{(p)}[\gamma](\nu, \nu),$$

corresponding to the kinetic energy associated with the material mass density, and into the kinetic energy of the dislocation density. By construction, the metric $g$ is constant in $\gamma$, that is

$$DG_{\gamma}[\nu](\eta) = 0 \quad \forall \eta \in T_\gamma \mathcal{V}(M; \mathbb{R}^3), \quad \gamma \in \mathcal{V}(M; \mathbb{R}^3).$$

Therefore, the corresponding Euler's equations yield

$$\mathcal{G}_{\gamma}[\gamma(\eta)](\dot{\gamma}(\eta), \eta) = 0, \quad \forall \eta \in T_\gamma \mathcal{V}(M; \mathbb{R}^3)$$

as weak equations of motion. The geodesics of $\mathcal{G}_{\gamma}$ are analogously to elasticity straight line segments, cf. [4, 6]. An inertial motion follows by definition the geodesics of $\mathcal{G}_{\gamma}$. A motion under non-vanishing forces will deviate from these geodesics.

5 The Principle of Virtual Work

In our setting, a work functional on the space of generalised configurations $\mathcal{V}(M; \mathbb{R}^3)$ is understood to be a continuous linear functional

$$F : T\mathcal{V}(M; \mathbb{R}^3) \equiv \mathcal{V}(M; \mathbb{R}^3) \times \Omega^1(M; \mathbb{R}^3) \rightarrow \mathbb{R},$$

on the tangent bundle $T\mathcal{V}(M; \mathbb{R}^3)$. We assume that for each configuration $\gamma \in \mathcal{V}(M; \mathbb{R}^3)$ the functional $F$ admits an integral representation with respect to the metric $g$ given in (2), such that

$$F[\gamma](\eta) = \int_M \langle \alpha[\gamma], \eta \rangle \mu \quad \forall \eta \in T_\gamma \mathcal{V}(M; \mathbb{R}^3).$$

(12)

The constitutive law of the continuum $M$ is encoded in the functional dependence of the integral kernel $\alpha[\gamma] \in \Omega^1(M; \mathbb{R}^3)$ on the configuration $\gamma$. This dependence will, in
general, be non-linear and possibly also non-local. More precisely, the integral kernel \( \alpha \) may be thought of as a smooth section into the tangent bundle \( TV(M; \mathbb{R}^3) \), where each \( \alpha[\gamma] \) is identified with its principal part. The one-form \( \alpha \) will be called stress form; in classical elasticity, \( \alpha \) corresponds to the first Piola-Kirchhoff stress tensor, cf. [5, 16].

For each \( \gamma \in \mathcal{V}(M; \mathbb{R}^3) \), the Helmholtz decomposition of \( \alpha[\gamma] \) reads

\[
\alpha[\gamma] = dh[\gamma] + \tau[\gamma],
\]

where \( dh[\gamma] \in d\Omega^0(M; \mathbb{R}^3) \) is a gradient and \( \tau[\gamma] \in \mathcal{D}(M; \mathbb{R}^3) \) is divergence-free. The decompositions are understood with respect to a fixed reference metric \( g \). Writing \( \eta = du + v \), the orthogonality of the Helmholtz decomposition implies

\[
\mathcal{G}(\alpha[\gamma], \eta) = \mathcal{G}(dh[\gamma], du) + \mathcal{G}(\tau[\gamma], v).
\]

Therefore, for each generalised configuration \( \gamma \in \mathcal{V}(M; \mathbb{R}^3) \), the work functional \( F \) splits into an elastic part \( F^{(e)} \) and a plastic part \( F^{(p)} \), i.e.

\[
F[\gamma](\eta) = F^{(e)}[\gamma](du) + F^{(p)}[\gamma](v) \quad \forall \eta = du + v \in T\gamma \mathcal{V}(M; \mathbb{R}^3).
\]

The elastic part is given by

\[
F^{(e)}[\gamma](du) := \int_M <dh[\gamma], du> \mu \quad \forall du \in d\Omega^0(M; \mathbb{R}^3),
\]

and the plastic part by

\[
F^{(p)}[\gamma](v) := \int_M <\tau[\gamma], v> \mu \quad \forall v \in \mathcal{D}(M; \mathbb{R}^3).
\]

Since the Helmholtz decomposition is orthogonal,

\[
F = F^{(e)} \iff \alpha[\gamma] = dh[\gamma] \quad \forall \gamma \in \mathcal{V}(M; \mathbb{R}^3).
\]

It was first observed in [3] that in pure elasticity, only the exact part \( dh[\gamma] \) of the stress form \( \alpha[\gamma] \) contributes to the work functional. In fact, \( F^{(e)} \) is the well-known work functional of elasticity, cf. [1, 7, 14]. The work functional (12) thus becomes a natural generalisation of the notion of work in classical elasticity.

Notice that both components \( dh[\gamma] \) and \( \tau[\gamma] \) of the stress form \( \alpha[\gamma] = dh[\gamma] + \tau[\gamma] \) will, in general, depend on the integrable part \( dj \) as well as the plastic part \( \beta \in \mathcal{D}(M; \mathbb{R}^3) \) of \( \gamma = dj + \beta \). From the elastic point of view, \( \tau \) marks a gauge freedom, cf. [5]. Hence, the choice of \( \tau \) describes the plastic part in view.

Next, we implement the work functional (12) in the d'Alembert principle of virtual work. According to [13], an exterior force acting on a general mechanical system is given by a horizontal one-form on the tangent manifold of the corresponding configuration space.
Recall that, using the tangent map \( T\tau_V : T^2V(M; \mathbb{R}^3) \rightarrow TV(M; \mathbb{R}^3) \) of the canonical projection \( \tau_V \), a vector field \( \mathcal{V} \) on \( TV(M; \mathbb{R}^3) \) is by definition *vertical* iff \( T\tau_V(\mathcal{V}) = 0 \). A one-form \( \mathcal{F} \) on \( TV(M; \mathbb{R}^3) \) is *horizontal* iff \( \mathcal{F}(\mathcal{V}) = 0 \) for all vertical vector fields \( \mathcal{V} \). Thus, an exterior force in the above sense acting on dislocated material is given by a horizontal one-form \( \mathcal{F} \) on \( TV(M; \mathbb{R}^3) \).

If \( \mathcal{V} \) is a vertical vector field and \( \Omega \) is the symplectic two-form defined in (10), then

\[
\Omega(\mathcal{V}, \mathcal{Z}) = -G_V(\gamma)(\mathcal{V}, T\tau_V \mathcal{Z}) \quad \forall \mathcal{Z} \in \Gamma(T^2V(M; \mathbb{R}^3)).
\]

Therefore, the induced one-form \( \iota_{\mathcal{V}} \Omega \) given by

\[
\iota_{\mathcal{V}} \Omega(\mathcal{Z}) := \Omega(\mathcal{V}, \mathcal{Z}) \quad \forall \mathcal{Z} \in \Gamma(T^2V(M; \mathbb{R}^3))
\]

is horizontal\(^2\). On the other hand, using the tangent map \( T\tau_V \) of the canonical projection \( \tau_V \), the work functional \( F \) defined in (12) induces an exterior work one-form \( \mathcal{F} \) in the above sense by setting

\[
\mathcal{F} := (T\tau_V)^*F. \quad (17)
\]

Due to the pull-back construction, \( \mathcal{F} \) is horizontal. Given the kinetic energy functional \( \mathcal{E} \) and an exterior work one-form (17), the d’Alembert principle of virtual work now states that the Euler vector field \( \mathcal{X} \) is determined by the equation

\[
d\mathcal{E}(\mathcal{Z}) - \iota_{\mathcal{X}} \Omega(\mathcal{Z}) = (T\tau_V)^*F(\mathcal{Z}) \quad \forall \mathcal{Z} \in \Gamma(T^2V(M; \mathbb{R}^3)). \quad (18)
\]

### 6 The Equations of Motion

In order to formulate a dynamics on our configuration space \( V(M; \mathbb{R}^3) \), consider a motion given by a smooth curve

\[
\gamma : \mathbb{R} \rightarrow V(M; \mathbb{R}^3), \quad t \mapsto \gamma(t).
\]

Using the exterior work functional (17), the curve \( \gamma(t) \) describes a motion subject to the d’Alembert principle of virtual work (18), if it satisfies the weak equations of motion

\[
G_V[\gamma(t)]=F[\gamma(t)] \quad \forall \eta \in \Omega^1(M; \mathbb{R}^3). \quad (19)
\]

According to Helmholtz, each \( \gamma(t), t \in \mathbb{R} \) decomposes into \( \gamma(t) = d\eta(t) + \beta(t) \). The orthogonality of the splittings of the work functional \( F = F^v + F^p \) and the metric

\(^2\) In the case where \( \Omega \) is regular, the converse also holds true: for any horizontal one-form \( \mathcal{F} \), there is a vertical vector field \( \mathcal{V}_\mathcal{F} \) such that \( \mathcal{F} = \iota_{\mathcal{V}_\mathcal{F}} \Omega \).
Given in Definition 4.1, respectively, implies that (19) is equivalent to
the system of equations

\[ G^c_v[\gamma(t)](\ddot{d}(t), du) = F^c(\gamma(t))(du) \quad \forall du \in d\Omega^0(M; \mathbb{R}^3) \] (20)

and

\[ G^p_v[\gamma(t)](\ddot{\beta}(t), v) = F^p(\gamma(t))(v) \quad \forall v \in D(M; \mathbb{R}^3). \] (21)

Thus, the dynamical equations derived from the principle of virtual work split into an elastic part (20) and into a plastic part (21). In absence of all external volume and surface forces, the equations of motion\(^3\) induced by (20) and (21) are given in the following theorem.

**Theorem 6.1** Let \( \alpha[\gamma] = dh[\gamma] + \tau[\gamma] \) be the Helmholtz decomposition of a stress form for a dislocated material. Then the equations of motion are given by

\[
\begin{align*}
\rho \dddot{y}(t) &= \Delta h[\gamma(t)] \\
\sigma \dddot{\beta}(t) &= \tau[\gamma(t)]
\end{align*}
\]

where \( \gamma(t) = \ddot{d}(t) + \beta(t) \) is the Helmholtz decomposition of \( \gamma(t) \) and \( \Delta := \delta d \) is the Laplace operator on functions in \( C^\infty(M; \mathbb{R}^3) \).

The first equation in Theorem 6.1 is nothing but the well-known equation of motion in elasticity: since \( \delta \tau[\gamma] = 0 \), the divergence of the stress form \( \alpha[\gamma] \) corresponding to the first Piola-Kirchhoff stress tensor can be represented as the Laplace operator on functions, i.e. \( \delta \alpha[\gamma] = \Delta h[\gamma] \). The second one is an evolution equation for the non-integrable parts of the deformation \( \gamma(t) \). The equations of motion are coupled via the Helmholtz decomposition. The motion of dislocations may, in general, be accompanied by dissipative effects, cf. [11].

In a static setting, \( \gamma \in \mathcal{V}(M; \mathbb{R}^3) \) is an equilibrium configuration if and only if

\[ F[\gamma](\eta) = 0 \quad \forall \eta \in T_\gamma \mathcal{V}(M; \mathbb{R}^3) \]

which according to (14) is equivalent to

\[ F^c(\gamma)(du) = 0 \quad \forall du \in d\Omega^0(M; \mathbb{R}^3) \quad \text{and} \quad F^p(\gamma)(v) = 0 \quad \forall v \in D(M; \mathbb{R}^3). \]

\(^3\)The equivalence of the weak equations and the strong equations follow from the fact, that the space of smooth differential forms is dense in an appropriate \( L^2 \)-completion, cf. [17].
The second Piola-Kirchhoff stress tensor $S[\gamma]$ associated with the stress form $\alpha[\gamma] \in \Omega^1(M; \mathbb{R}^3)$, $\gamma \in \mathcal{V}(M; \mathbb{R}^3)$ is given by

$$S[\gamma](X,Y) := \langle \alpha[\gamma](X), \gamma(Y) \rangle_{\mathbb{R}^3}, \quad X,Y \in \Gamma(TM).$$

In pure elasticity, there is a gauge freedom in choosing the stress form. Since only the integrable part $dh[\gamma]$ of a stress form $\alpha[\gamma]$ contributes to the work functional of elasticity $F^{(e)}$, any stress form $\tilde{\alpha}[\gamma] = \alpha[\gamma] + \xi[\gamma]$ with arbitrary $\xi[\gamma] \in \mathcal{D}(M; \mathbb{R}^3)$ will give the same work functional $F^{(e)}$ and hence determine the same dynamics of the system, cf. [3]. In particular, one may chose $\xi[\gamma]$ such that the stress tensor $\tilde{S}$ corresponding to $\tilde{\alpha}[\gamma]$ is symmetric, cf. [16].

In the dislocated case, this gauge freedom is lost. Since the divergence-free part $\tau$ of the stress form $\alpha$ appears explicitly in the principle of virtual work (19), the stress tensor may not chosen to be symmetric. The concept of decomposing configurations $\gamma \in \mathcal{V}(M; \mathbb{R}^3)$ and stress forms $\alpha[\gamma] \in \Omega^1(M; \mathbb{R}^3)$, $\gamma \in \mathcal{V}(M; \mathbb{R}^3)$ is completely analogous to the concept of strain spaces and stress spaces in [10]. The integrable part of the deformation is the dual quantity to the integrable part of the stress, the non-integrable part of the deformation is the dual quantity to the non-integrable part of the stress.

References


