Calculating p-adic orbital integrals on GSp(4) via a family of special subgroups

Michael Schröder

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Universität Mannheim
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Particularly in applications of the trace formula intriguing problems are already posed by the local orbital integrals

$$O_U^G(f) = \int_{G(F) \setminus G(F)} f(g^{-1} s g) \, dg.$$ 

For $G = GSp(4)$ and a semisimple element $s$ in $G(F)$ which is regular we propose in this paper a two-step method for calculating them explicitly. It is based on subgroups $H_C$ of $G$ depending only on the stable conjugacy class $C$ of $s$, and possessing a hypothetical codiscreteness property which we prove and apply.

0. Notation: Unless otherwise specified $F$ is a nonarchimedean local field with uniformizing element $\pi$, ring of integers $\mathcal{O}_F$ and residue field $k(F)$. We write $U(F)$ for the set of units in $\mathcal{O}_F$.

Let $I$ be the involution on $M(2n, F)$, the $2n$ by $2n$ matrices with coefficients in $F$.

$$I(g) = J^{-1} \cdot g \cdot J$$

with

$$J = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}.$$ 

The group $GSp(2n, F)$ of symplectic similitudes is the set of all $g$ in $M(2n, F)$ such that $I(g) \cdot g = \mu(g) \cdot J$ or equivalently such that $I(g) \cdot g = \mu(g) \cdot E_{2n}$.

1. The groups $H_C$: Let $F$ be any perfect field. A method to classify the stable conjugacy classes $C$ of maximal $F$-tori in $GSp(2n)$, by which $H_C$ are parametrized, dates back to A. Weil. Let $T$ be a torus in $C$ and $s$ in $T(F)$ regular. The centralizer $C(s)$ of $s$ in $M(2n, F)$ is isomorphic to the algebra $F[s]$. Now $F[s]$ is isomorphic to the direct sum $\bigoplus_{E \in E_F} E$ of the extension fields $E$ of $F$ defined by the irreducible factors of the characteristic polynomial of $s$. The image $\tau_C$ of $T(F)$ in $E_C$ consists of all $x$ in $E_C$ with $I(x)x = \mu(x)1$. One is thus led to study the action of $I$ on the elements of $F$.

By [S, Kapitel 5] each $E$ in $F$ either belongs to a pair of factors $(E, E')$ which $I$ interchanges, i.e. with $I(E) = E'$, or $I$ restricts to a nontrivial involution $\sigma_E$ on $E$. The first case gives rise to tori in Levi factors coming from general linear groups. In the second case we obtain tori of unitary groups. The group $H_C$ can be thought of as intersection with $GSp(2n)$ of the smallest product of general linear groups which contains representatives of all conjugacy classes in $C$. Thus it decomposes in two factors uniquely determined by $C$ only: One factor is constructed from the unitary data of the second case, the other from the Levi factors of the first case above. No new insights are required for the part of general linear type.

We thus assume that each factor in $F$ is of unitary type: $F = F_{\text{unitary}}$. For any $E$ in $F$ let $E^+$ and $E^-$ be the $(+1)$- and $(-1)$-eigenspaces of $\sigma_E$ respectively. Decompose $E_C = E^+_C \oplus E^-_C$ accordingly.

For any invertible $a = (a_E)$ in $E_C$ define the symplectic form $B_C(a)$ on the $F$-space $E_C$ by

$$B_C(a) ((x_E), (y_E)) = \sum_{E \in F} \text{tr}_{E/K} \left( x_E \cdot a_E \cdot \sigma_E (y_E) \right).$$

Conjugating by $\text{diag}(\ldots, \text{diag}(1,b_E), \ldots)$ with $b = (b_E)$ invertible in $E^+_C$ transforms $GSp(\mathcal{E}_C, B(a))$ into $GSp(\mathcal{E}_C, B(ab))$. Therefore

$$H_C(F) = GSp(\mathcal{F}_C, B_C(ab)) \cap \bigoplus \{ GL_{E^+}(E) : E \in F \}$$

is independent of the form $B(a)$ and defines the algebraic $F$-subgroup $H_C$ of $GSp(2n)$. We embed $\mathcal{E}_C$ in $\text{End}_F(\mathcal{E}_C)$ mapping $a$ to the multiplication $\vartheta(a)$ by $a$ on the left. The intersection of $I(\mathcal{E}_C)$
with each group $GSp(E, B(a))$ is $\xi(\tau_C)$, so that $\xi(\tau_C)$ defines a subtorus $T_C$ of $H_C$ in the stable conjugacy class $C$. For all $b = (b_E)$ invertible in $F^2$ we let

$$T_C(b) = \left( \text{Int diag} \left( \begin{array}{cc} 1 & 0 \\ 0 & b_E \end{array} \right) \right) (T_C).$$

The $T_C(b)$ account for all conjugacy classes of tori in $C$. Furthermore, $T_C$ is conjugate to a $T_C(b)$ in $GSp(2n, F)$ if and only if $T_C$ is conjugate to $T_C(b)$ in $H_C(F)$.

2. The $GSp(4)$ results: The stable conjugacy classes of maximal $F$-tori in $GSp(4)$ with unitary parts were classified in [S, Kapitel 6]. They are the tori of type $T_{3A}$ and the basic tori which, in fact, both have no parts of general linear type. This is the situation where we expect our codiscreteness results to hold in general.

(2.1) Tori of type $T_{3A}$: Such a torus is characterized by a pair $\xi = (E, L)$ of quadratic extensions of $F$ and by definition is isomorphic to the $F$-subtorus

$$\tau_{E,L} = \left\{ (x, y) \in E^* \times L^* : N_{E/F}(x) = N_{L/F}(y) \right\}$$

of the $F$-torus $E^* \times L^*$. Fix normalized primitive elements $\sqrt{A}$ of $E$ and $\sqrt{B}$ of $L$ over $F$, so that $A$ and $B$ are representatives in $F$ of $F^*/(F^*)^2$ which both have orders 0 or 1. We take $1_E, 1_L, \sqrt{A}, \sqrt{B}$ as symplectic orthonormal basis so that $H(F) = H_{\xi}(F)$ consists of all matrices

$$[h, h'] = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \quad \text{with} \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

in $GL(E) \times GL(L)$ which satisfy the symplecticity condition $det h = det h'$. We will prove

(2.2) Theorem: The set $H(F) \backslash GSp(4, F)/GSp(4, \mathcal{O}_F)$ is discrete, representatives are

$$g(\gamma) = \begin{pmatrix} E_2 & \gamma W \\ 0 & E_2 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with either $\gamma = 0$ or $\gamma = \pi^{-\ell}$ and $\ell > 0$ any natural number.

(2.3) Basic tori: Such a torus is characterized by a pair $\xi = (E, \sigma_E)$ consisting of an extension $E$ over $F$ of degree four which has a nontrivial involution $\sigma_E$. Let $\sqrt{A}$ be a normalized primitive element over $F$ of the fixed field $E^+$ of $\sigma_E$. Then $H_{\xi} \cong \{ g \in GL(2, E^+) : det g \in F^* \}$. By [S, A.19.8] it can be realized as the subgroup of $GSp(4, F)$ of all matrices

$$\begin{pmatrix} a_1 & a_2 A^{-1} & b_1 & b_2 \\ a_2 & a_1 & b_2 & b_1 A \\ c_1 & c_2 A^{-1} & d_1 & d_2 \\ c_2 A^{-1} & c_1 A^{-1} & d_2 A^{-1} & d_1 \end{pmatrix}$$

Actually, this is the embedding used by Prof. Weissauer in his proof of the fundamental lemma. By definition, the basic torus to $\xi$ is $\tau_{\xi}(\ell) = \{ x \in E^* : N_{E/F}(x) \in F^* \}$. We will prove

(2.4) Theorem: The set $H_{\xi} = H_{\xi}(F) \backslash GSp(4, F)/GSp(4, \mathcal{O}_F)$ is discrete. For all $\ell \geq 1$ let

$$g(\ell) = \begin{pmatrix} E_2 & S(\ell) \\ 0 & E_2 \end{pmatrix} \quad \text{with} \quad S(\ell) = \begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & 0 \end{pmatrix}.$$
3. A technique for calculating orbital integrals: Let \( s \) be a regular, \( F \)-rational element of the torus \( T_c(b) \). Let \( K = GSp(2n, OF) \). Assume \( H_C \cap GSp(2n, F) \) is discrete, thus countable. For any Hecke operator \( f \) on \( GSp(2n, F) \) one has by [Wa I, p.477, A 1.2] and [KoGL, p.361f]

\[
O^G_{sGSp(2n)}(f) \overset{\text{def}}{=} \int_{T_c(b) \backslash GSp(2n)} f(g^{-1}sg)dg
\]

\[
= \sum_{x \in H_C \cap GSp(2n)/K} \int_{T_c(b) \backslash H_C} (f \circ \text{Ad}^{-1})(h^{-1}sh)dh
\]

where we identified their \( F \)-rational points and measures are suitably normalized. The support of \( f \circ \text{Ad}^{-1} \) is \( \text{supp}(f) = H_C(F) \cap (\text{supp}(f)) \cdot x^{-1} \).

We want to show next that this is a tool for calculating orbital integrals for the group \( GSp(4) \).

4. Calculating the \( GSp(4) \)-orbital integral \( O_s(T(\pi)) \) for \( s \) in a torus of type \( T_{3A} \): Let \( F \) be a local field with odd residue characteristic and \( G = GSp(4) \). Let \( \pi \) be the Hecke operator on \( GSp(4) \) defined as characteristic function of the double coset \( K \cdot \text{diag}(E_2, \pi E_2) \cdot K \) with \( K = GSp(4, OF) \). This operator is closely connected with the problem of counting points mod \( p \) of the Shimura variety to \( G \) by the trace formula.

(4.1) Embeddings of tori of type \( T_{3A} \): In the local case the first cohomology group of the torus \( T_{E,L} \) of type \( T_{3A} \) is by class field theory trivial for \( E \neq L \) and cyclic of order two for \( E = L \). Representatives of the conjugacy classes of \( F \)-embeddings of \( T_{E,L} \) into \( GSp(4) \) taking their values in \( H \) were determined explicitly in [5, §II B]. We fix the \( F \)-rational, semisimple element \( s \) in the image of \( T_{E,L} \) under any of these, so that

\[
s = [s_E, s_L] = \left[ \begin{array}{cc} a & bD^{-1}A \\ b & a' \end{array} \right], \left[ \begin{array}{cc} a' & bB \\ b' & a' \end{array} \right] \sim F \left[ \begin{array}{cc} \lambda & \lambda' \\ \mu & \mu' \end{array} \right]
\]

where \( D = 1 \) in the stable case \( E \neq L \) and where \( D \in \{1, \sqrt{-1}\} \equiv F^*/N_{E/F}(E^*) \) in the instable case \( E = L \). Our aim is to prove

(4.2) Theorem: Let \( F \) be a local field with odd residue characteristic and let \( s \) be regular. The \( GSp(4) \)-orbital integral \( O_s(T(\pi)) \) is nonzero only if the similarity factor \( \mu(s) \) of \( s \) has order one in \( F \) and \( E, L \) are both ramified over \( F \).

Then in the stable case \( E \neq L \)

\[
O_s(T(\pi)) \overset{\text{def}}{=} \int_{T \backslash GSp(4)} T(\pi)(g^{-1}sg)dg \left( \begin{array}{c} \mu_{GSp(4)} \\ \mu_T \end{array} \right)(g) = 2 \cdot \frac{\text{vol}_G(GSp(4, OF))}{\text{vol}_T(T(OF))}.
\]

In the instable case \( E = L \) let \( \delta_D(s) = 1 \) if \(-bD/b' \) is a quadratic residue modulo \( \pi OF \) and \( \delta_D(s) = 0 \) otherwise. Then

\[
O_s(T(\pi)) = \frac{\text{vol}_G(GSp(4, OF))}{\text{vol}_T(T(OF))} \left( 1 + \frac{2 \cdot \delta_D(s)}{[\lambda + \mu] - (\lambda' + \mu')] \cdot \xi_F(1) \right)
\]

where \( N = \text{ord}_F((\lambda + \mu) - (\lambda' + \mu')) \) and \( \xi_F(\ell) = 1/(1 - q^{-\ell}) \) is the zeta function of \( F \) evaluated at \( \ell \).

Remarks: In the instable case \( E = L \) our calculation identifies the instable contribution from the regular elements in tori of type \( T_{3A} \) to the trace formula "evaluated for \( T(\pi) \)." The \( \kappa \)-orbital integral to \( s \) on \( GSp(4) \) is in this case up to a sign the difference of the orbital integrals to the two conjugates of \( s \). Thus we get \( \Delta(s) \cdot O^\kappa_s(T(\pi)) = O^\kappa_s(T(0, \pi)) \) with transfer factor \( \Delta(s) = \pm |\lambda|^{1/2} \cdot (|\lambda' - 1| \cdot (\lambda' / \lambda - 1) \cdot (\mu / \lambda - 1) \cdot (\lambda' / \lambda - 1)]^{1/2} \cdot \xi_F(N)/\xi_F(1) = \pm (|\lambda + \mu| - (\lambda' + \mu')) \cdot \xi_F(N)/\xi_F(1). \)
(4.3) The operators $T(\ell, \pi)$ and the groups $H(\ell)$: Define for all integers $\ell \geq 0$

(2) $z(0) = E_4$, $z(\ell) = \begin{pmatrix} 0 & \pi^{-\ell}E_2 \\ E_2 & 0 \end{pmatrix} g(\pi^{-\ell}) = \begin{pmatrix} 0 & \pi^{-\ell}e_2 \\ e_2 & \pi^{-\ell}w \end{pmatrix}$, $W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\ell \geq 1$,

(2) $H(\ell) = H(F) \cap z(\ell) \cdot K \cdot z(\ell)^{-1}$,

(4) $T(\ell, \pi) = T(\pi) \circ (\text{Ad} z(\ell)^{-1})|_{H(F)} \in \mathcal{H}(H(F), H(\ell))$.

The family of all $z(\ell)$ is again a system of representatives for $H(F) \backslash GSp(4, F)/K$. We claim

(4.3.1) Proposition: For any $\ell \geq 0$ the support of $T(\ell, \pi)$ is $H(\ell) \cdot \text{[diag}(1, \pi), \text{diag}(\pi, 1)] : H(\ell)$ and $\text{supp}(T(\ell, \pi)) \subseteq \text{supp}(T(\ell', \pi)) \subseteq \text{supp}(T(0, \pi))$ for all $\ell > \ell'$.

(4.3.2) Lemma: Let $pr$ be the projection of $H(F)$ on its first $GL(2)$-factor. Then for all $\ell \geq 0$ the sequence $1 \longrightarrow \{E_2\} \times \Gamma(\ell) \longrightarrow H(\ell) \longrightarrow GL(2, \mathcal{O}_F) \longrightarrow 1$ is exact, where $\Gamma(\ell)$ is the principal congruence subgroup of $SL(2, \mathcal{O}_F)$ of level $\pi^\ell$.

Furthermore $H(\ell) = \{[X, Y] \in H(\mathcal{O}_F) : X \equiv Y^w (\text{mod} \gamma^{-1}\mathcal{O}_F)\}$ where $Y^w = (\text{Int} w)(Y)$ and $w = E_{12} + E_{21}$.

(4.3.3) Symmetrization by the automorphism $1 \times \Phi$ of $H$: For $W = E_{12} + E_{21}$ let

(5) $(1 \times \Phi)[[h, h']] = (\text{Ad} ([E_2, W]))[[h, h']] = [h, (\text{Ad} W)(h')] = [h, \Phi(h')]$.

Then $H_\Phi(\ell) = (1 \times \Phi)(H(\ell)) = \{[X, Y] \in H(\mathcal{O}_F) : X \equiv Y^w (\text{mod} \gamma^{-1}\mathcal{O}_F)\}$ and the support of the pullback of $T(\ell, \pi)$ by $1 \times \Phi$ is $H_{\Phi}(\ell) \cdot [\text{diag}(1, \pi), \text{diag}(\pi, 1)] : H_{\Phi}(\ell)$.

A straightforward calculation shows (4.3.2). To prove (4.3.1) we first indicate a general strategy to determine the $H(\ell)$-double cosets $e$ in the support of the pullback $f \circ (\text{Ad} z(\ell)^{-1})$ to $H(F)$ of a Hecke operator $f$. Choose a representative of $e$ whose first $GL(2)$-component is a diagonal matrix $\text{diag}(a_1, d_1)$ with pure $\pi$-powers $a_1, d_1$ and $\text{ord } a_1 \geq \text{ord } d_1$. We have to decide when $Y_{\ell, S} = z(\ell)^{-1} \cdot [\text{diag}(a_1, d_1), \text{diag}(d_1, a_1) \cdot S] \cdot z(\ell)$ is in the support of $f$, where $S$ is chosen in $SL(2, F)/\Gamma(\ell)$.

Using the filtration $\Gamma(\ell + 1) \subseteq \Gamma(\ell) \subseteq \Gamma(0)$ one should deal with this problem iteratively, starting with $\ell = 0$. For $\ell$ fixed, one decides in a first step for which parameters $Y_{\ell, S}$ has entries in $\mathcal{O}_F$. Only for these one determines then in a second step the elementary divisors.

Let $f = T(\pi)$. Then $a_1 = 1, d_1 = \pi$ imply that $Y_{\ell, S}$ has entries in $\mathcal{O}_F$ only if $S$ is in $\Gamma(\ell)$.

(4.4) Necessary conditions on $s$: The similarity factor $\mu(s) = a^2 - b^2A = (a')^2 - (b')^2B$ of $s$ has by Hensel's lemma the same order one as $\mu(T(\pi))$ only if $\text{ord } A = \text{ord } B = 1$, if $\text{ord } a, \text{ord } a' \geq 1$, and if $b, b'$ are both units. Especially then, $E$ and $L$ are ramified over $F$.

We use an iterative procedure based on the support filtration (4.3.1) to calculate the orbital integrals $O_{\ell}(T(\ell, \pi))$, the key step being (4.6).

(4.5) The parameter set $N(E, L)$: We have $E^* / pr(T(F)) \cong N(E, L)$, where

$$N(E, L) = \{ [E_4, \pi] : E = L \} \text{ with } x = [x_B, x_L] = \begin{bmatrix} D^{-1}A & 0 \\ 0 & A \end{bmatrix},$$

since by construction $\mu(T(F)) = N_E(F)^* \cap N_L(F)^*(L^*)$.

(4.6) Proposition: In the case $E = L$ the support of $O_H^E(T(0, \pi))$ is

$$\{ h \in H(F) : T(0, \pi)(h^{-1}sh) \neq 0 \} \supseteq T(\mathcal{O}_F) \backslash H(\mathcal{O}_F) = \{ T(\mathcal{O}_F) : h : h \in H(\mathcal{O}_F) \}.$$
(4.7) Proposition: For \( \alpha = [x_E, x_L] \) in \( \mathcal{N}(E, L) \) and \( \ell \geq 1 \) let \( R(\alpha, \ell, s) \) be the number of \( Y \) in \( SL(2, \mathcal{O}_F/\pi^{\ell} \mathcal{O}_F) \) such that \( [SE, y^{-1}(a_L^{-1}sL\alpha_L)y] \) is in the support of \( T(\ell, \pi) \). Then

\[
O_s(T(\ell, \pi)) = \frac{\text{vol}_{\ell}(H(\ell))}{\text{vol}_{\ell}(T(\mathcal{O}_F))} \sum_{\alpha \in \mathcal{N}(E, L)} R(\alpha, \ell, s),
\]

\[
O_s(T(\pi)) = \frac{\text{vol}_{\ell}(GSp(4, \mathcal{O}_F))}{\text{vol}_{\ell}(T(\mathcal{O}_F))} \left( [E : E] + \sum_{\alpha \in \mathcal{N}(E, L)} \sum_{s > 0} R(\alpha, \ell, s) \right).
\]

The elements \([1, y] \) with \( y \) in \( SL(2, \mathcal{O}_F/\pi^{\ell} \mathcal{O}_F) \), which we consider as section in \( M(2, \mathcal{O}_F) \) for \( SL(2, \mathcal{O}_F)/H(\ell) \), are representatives for \( H(\mathcal{O}_F)/H(\ell) \). So (4.7) follows by (4.6) and (331).

Proof of (4.6): Our argument is based on the fact that \( g_n = \text{diag}(1, \pi^n) \) with \( n \geq 0 \) are representatives of \( \tau(F) \backslash GL(2, F)/GL(2, \mathcal{O}_F) \), where \( \tau \) is any of the tori \( E^*, L^* \) or \( x_L^{-1}L^*x_L \).

Decompose \( \mathcal{E}^* \) in the form \( \mathcal{E}^* = \mathcal{E}^*aE \) and choose \( tL \) in \( L^* \) such that \( \det tL = \det \mathcal{E}^* \). By construction \( y = (tL^{-1}sL^{-1})^{-1} \) is in \( SL(2, \mathcal{O}_F) \) and \( \tau(F) \backslash GL(2, \mathcal{O}_F) \). Then \( \tau \) is a unit, so that the entries of \( \mathcal{E}^*sl \) are in \( \mathcal{O}_F \).

Then \( \tau \) is a unit, so that the entries of \( \mathcal{E}^*sl \) are in \( \mathcal{O}_F \).

(4.8) Characterizing \( R(\alpha, \ell, s) \) by congruence conditions: Let \( P = \text{diag}(1, \pi) \), so that \( sE = hEP \) for \( hE \) in \( GL(2, \mathcal{O}_F) \). Let \( Y \) be in \( SL(2, \mathcal{O}_F/\pi^{\ell} \mathcal{O}_F) \), taken as set of representatives for \( SL(2, \mathcal{O}_F)/\tau(F) \) in \( M(2, \mathcal{O}_F) \). Then \( [SE, y^{-1}(a_L^{-1}sL\alpha_L)y] \) is in the support of \( T(\ell, \pi) \).

The order conditions hold only for the elements \( (\alpha, \beta, \delta) \) with \( \gamma = \frac{\omega \beta - 1}{\delta} \) and \( \omega \in \{ \pi, \ldots, \pi^{\ell-1} \} \), \( \beta \in \{ 1, \pi, \ldots, \pi^{\ell-1} \} \), \( \delta \in k(F)^* \) such that \( \tau = \pi^{\ell-1}k(F)^* \).

They are defined by the intersection of three quadrics in the affine space of dimension four over \( \mathcal{O}_F/\pi^{\ell} \mathcal{O}_F \): In the case \( \alpha = E \) multiply \( Y \), with entries denoted as in (7), by \( \omega - \delta \sqrt{B} \) on the left and let \( \Delta(Y) = \omega^2 - \delta^2 B, z(Y) = \omega - \delta B \). For \( \alpha = x \) multiply by \( x^{-1}(\omega - \delta(A/B) \sqrt{B})xL \) and let \( \Delta(X) = \omega^2 - \delta^2(A/B) \cdot A, z(X) = \omega - \delta(B/A) \cdot A \). Then

\[
\Phi(Y^{-1}sL_Y) = \left( \begin{array}{cc} a' + zLb' & b' \Delta_L \\ b'(B - zL) \Delta_L & a' - zLb' \end{array} \right),
\]

\[
\Phi(Y^{-1}(a_L^{-1}sLxL)_Y) = \left( \begin{array}{cc} a' + zLb' & b' \Delta_L \\ b'(A - A^{-1}BzL) \Delta_L & a' - zLb' \end{array} \right).
\]

Their entries have orders as in (6) only if \( \text{ord } \Delta_a = 1 \) and \( \text{ord } z_a \geq 1 \). These conditions characterize the elements \( Y(\omega, \beta, \delta) \).

By construction \( \Phi(Y^{-1}(a_L^{-1}sLxL)_Y)P^{-1} \) has entries in \( O_F \) for all \( Y = Y(\omega, \beta, \delta) \). Thus the following easily proved criterion applies to determine for which \( (\alpha, \beta, \delta) \) the \( \Gamma(\ell) \)-double cosets of \( \Phi(Y^{-1}(a_L^{-1}sLxL)_Y) \) and \( sE = hEP \) are equal.

(4.8.1) Lemma: Let \( g', g'' \) be in \( GL(2, \mathcal{O}_F) \). Then \( \Gamma(\ell) \cdot g' \cdot \Gamma(\ell) = \Gamma(\ell) \cdot g'' \cdot \Gamma(\ell) \) if and only if there is \( \beta \) in \( \pi^{\ell-1}k(F) \) such that

\[
g' \equiv g'' \left( \begin{array}{cc} 1 & \beta \\ 0 & 1 \end{array} \right) \pmod{\Gamma(\ell)}.
\]

5
For \( \alpha \equiv E_4 \) these congruences are equivalent to \( a \equiv a'(\mod \pi^4 \mathcal{O}_F) \), \( 0 \equiv z(\mod \pi^4 \mathcal{O}_F) \) and

\[
\Delta_0 = \frac{\Delta}{\pi} \equiv \frac{A}{\pi D} b \left( \mod \pi^4 \mathcal{O}_F \right), \quad \left( \frac{b}{b'} \right)^2 = \frac{A}{\pi} \equiv \frac{B - z^2}{\pi} \left( \mod \pi^4 \mathcal{O}_F \right).
\]

Especially then, the values \( \mod \pi^4 \mathcal{O}_F \) of \( \Delta_0 \), \( z \) are completely determined. Since \( \text{ord} \ z \geq \ell \) we obtain \( (b/b')^2 \equiv B/A(\mod \pi^4) \). So \( A = B \) by Hensel's lemma and since \( A \) and \( B \) are normalized.

Consequently \( (b/b')^2 \equiv 1(\mod \pi^4) \). In the case \( \alpha = z \) we obtain similar congruences. They, too, hold only for \( A = B \), contradicting the assumption \( A \neq B \).

We note that \( a \equiv a'(\mod \pi^4 \mathcal{O}_F) \) implies \( b^2 \equiv (b')^2(\mod \pi^4 \mathcal{O}_F) \) and thus \( b \equiv \epsilon b'(\mod \pi^4 \mathcal{O}_F) \) because of (4.4).

For a smooth affine variety \( V \) over \( \mathcal{O}_F \) the fibres of \( V(\mathcal{O}_F/\pi^{i+1} \mathcal{O}_F) \rightarrow V(\mathcal{O}_F/\pi^i \mathcal{O}_F) \) have cardinality \( \#k(F)^{\dim V} \) for all \( i \geq 1 \). The following result, proved by showing that the Jacobian has full rank, now completes the proof of (4.2)

(4.8.2) Lemma: For \( z \) in \( \pi \mathcal{O}_F \), \( \gamma \) in \( \pi U(F) \) and \( \Delta_0 \) in \( U(F) \) let \( Q_\gamma \) be the zero set of \( \Delta_0 = \pi^{-1}(\omega^2 - \delta^2 \gamma) \), \( z = \omega \gamma - \delta \beta \gamma \), \( 1 = \omega \beta - \beta \gamma \) in the fourdimensional affine space. Then \( Q_\gamma \) is a smooth variety over \( \mathcal{O}_F \) and

\[
\# \left\{ (\omega, \beta, \delta, \gamma) \in Q_\gamma(\pi^i \mathcal{O}_F/\pi^{i+1} \mathcal{O}_F) : \omega \equiv 0(\mod \pi \mathcal{O}_F) \right\} = \begin{cases} 2 \cdot (\#k(F))^{\ell} & \frac{\pi}{\Delta_0} \in U(F)^2 \\ \text{0} & \text{otherwise.} \end{cases}
\]

5. Proof of Theorem (2.2): Using the Iwasawa decomposition of \( GSp(4, F) \) one starts with representatives in the Borel subgroup of upper triangular matrices in \( GSp(4, F) \). Multiplying on the left by suitable upper triangular matrices in \( H(F) \) they can be modified to representatives of the form \( h(a, b, 0) \) in the Heisenberg subgroup of \( GSp(4, F) \) consisting of all matrices

\[
h(a, b, c) = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a & 1 \end{pmatrix}
\]

with \( a, b, c \) in \( F \). Since \( h(0, 0, c) \) is in \( H(F) \) for all \( c \) in \( F \), the relations

\[
h(a, b, 0) \cdot h(a', b', 0) = h(a + a', b + b', ab' - a'b), \quad h(0, 0, -c) \cdot h(a, b, c) = h(a, b, 0)
\]

show that one may in fact choose representatives \( g(a, b) = h(a, b, 0) \) with \( a, b \) in the \( k(F) \)-space \( \{ \ldots, \pi^2, \pi^{-1} \} \). Here we follow a suggestion of Prof. Weissauer for simplifying our original proof.

We reduce to pure \( \pi \)-powers \( a \) and \( b \). Let \( \alpha = \omega a \) and \( \beta = \omega b \) for units \( \omega \) und \( \omega \) in \( U(F) \). Then

\[
g^{-1}(\alpha, \beta) \left[ \begin{array}{cc} u \omega & 0 \\ 0 & 1 \end{array} \right], \left[ \begin{array}{cc} w & 0 \\ 0 & u \end{array} \right] g(a, b) = \text{diag}(uw, w, 1, u)
\]

is in \( GSp(4, \mathcal{O}_F) \) so that \( g(\alpha, \beta) \) and \( g(a, b) \) are in the same coset of \( H(F) \setminus GSp(4, F) / GSp(4, \mathcal{O}_F) \).

By the same reasoning we reduce further to representatives \( g(\gamma) = g(0, \gamma) \). In the case \( \text{ord} \ b \leq \text{ord} \ a \)

\[
g^{-1}(0, b) \left[ \begin{array}{cc} b^{-1} & 0 \\ 0 & b^{-1} \end{array} \right], \left[ \begin{array}{cc} a & b \\ 0 & a \end{array} \right] g(a, b) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ b^{-1} & ab^{-1} & 0 & 1 \\ 0 & b^{-1} & 1 & 0 \end{pmatrix}
\]

is an element of \( GSp(4, \mathcal{O}_F) \). For \( \text{ord} \ a < \text{ord} \ b \)

\[
g^{-1}(0, a) \left[ \begin{array}{cc} a^{-1} & 0 \\ 0 & a \end{array} \right], \left[ \begin{array}{cc} a & b \\ 0 & -a \end{array} \right] g(a, b) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ a^{-1} & 1 & 0 & a^{-1}b \\ 0 & 0 & -1 & -a^{-1} \end{pmatrix}
\]

is in \( GSp(4, \mathcal{O}_F) \).
To prove independence, assume that \( g(\alpha) \) and \( g(\beta) \) are in the same coset. Equivalently, there is \( h \) in \( H(F) \) such that \( hg(\alpha) \cdot GSp(4, O_F) = g(\beta) \cdot GSp(4, O_F) \). Taking images of this \( GSp(4, O_F) \)-coset under each element of a dual basis one obtains four equalities of ideals in \( O_F \). They translate into four conditions on the orders of the entries of \( hg(\alpha) \) and \( g(\beta) \). Distinguishing the cases \( \alpha \beta = 0 \) and \( \alpha \beta \neq 0 \) it follows easily that \( \alpha \) and \( \beta \) have the same orders and thus are in fact equal.

6. Proof of Theorem (2.4): One starts again with representatives in the Borel subgroup of upper triangular matrices in \( GSp(4, F) \). Their components in the Levi factor \( \{ diag(A, \lambda^A) : A \in GL(2, F), \lambda \in F^* \} \) can be reduced to matrices \( diag(g_\ell, g_\ell^{-1}) \), where \( g_\ell = diag(1, \pi^\ell) \) with \( \ell \geq 0 \) are representatives of \((E^+)^* \setminus GL(2, F)/GL(2, O_F)\). Because of

\[
\begin{pmatrix}
1 & 0 & b_1 & b_2 \\
0 & 1 & b_1A & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & x & y \\
0 & 1 & \pi^t & \pi^t y \\
0 & 0 & 0 & \pi^{-t}
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & a + b_1 & \pi^{-\ell} (b_2 + \pi^\ell y) \\
0 & 1 & b_2 + \pi^t y & \pi^t z + \pi^{-\ell} b_1 A \\
0 & 0 & 0 & \pi^{-t}
\end{pmatrix}
\]

we can choose representatives with \( y = z = 0 \). After multiplying from the right by a suitable unipotent matrix in \( GSp(4, O_F) \) we can assume \( x \in \langle \ldots, \pi^{-2}, \pi^{-1} \rangle \) and obtain the matrices \( g(\ell, x) \).

We now show that we can achieve \( \ell = 0 \). For \( \ell \geq 1 \)

\[
g(0, \pi^{-\ell})^{-1} \begin{pmatrix}
\pi^{-\ell} & 0 & 0 & 0 \\
0 & \pi^{-\ell} & 0 & 0 \\
1 & 0 & \pi^\ell & 0 \\
0 & A^{-1} & 0 & \pi^\ell
\end{pmatrix} g(\ell, 0) = 
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & \pi^\ell & 0 \\
0 & A^{-1} \pi^\ell & 0 & 1
\end{pmatrix}
\]

is in \( GSp(4, O_F) \). For \( x \neq 0 \) let \( z = -(1 + \pi^t)x^{-1} \). Then

\[
g(0, -x\pi^{-\ell})^{-1} \begin{pmatrix}
\pi^{-\ell} & 0 & 0 & 0 \\
0 & \pi^{-\ell} & 0 & 0 \\
z & 0 & \pi^\ell & 0 \\
0 & zA^{-1} & 0 & \pi^\ell
\end{pmatrix} g(\ell, x) = 
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
z & 0 & -1 & 0 \\
0 & zA^{-1} & 0 & 1
\end{pmatrix}
\]

is in \( GSp(4, O_F) \). We reduce to pure \( \pi \)-powers by the calculation

\[
g(0, x)^{-1} \begin{pmatrix}
E_2 & 0 \\
0 & u E_2
\end{pmatrix} g(0, y) = 
\begin{pmatrix}
E_2 & y - ux & 0 \\
0 & u E_2 & 0
\end{pmatrix}
\]

Eventually shows that representatives of \( \mathcal{H}_\ell \) are of the form \( E_4 = g(0, 0) \) and \( g(\ell) = g(0, \pi^{-\ell}) \) with \( \ell \geq 1 \) for \( \text{ord } A = 0 \), i.e. for \( E^+ \) unramified over \( F \), and \( g(\ell) \) with \( \ell \geq 1 \) for \( \text{ord } A = 1 \).

To check their independence is tedious, but straightforward given the method indicated in §5.

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8. References