Diese Arbeit bildet Teil der Dissertation des Verfassers.
EQUATIONALLY COMPACT ARTINIAN RINGS

By a Noetherian (Artinian) ring \( R = \langle R; +, -, 0, \cdot \rangle \) we mean an associative ring satisfying the ascending (descending) chain condition on left ideals. An arbitrary ring \( R \) is said to be \textit{equationally compact} if every system of ring polynomial equations with constants in \( R \) is simultaneously solvable in \( R \) provided every finite subset is. (The reader is referred to [21], [8], [13] and [14] for terminology and relevant results on equational compactness, and to [4] for unreferenced ring-theoretical results.) In this report a characterization of equationally compact Artinian rings is given - roughly speaking, these are the finite direct sums of finite rings and Prüfer groups; as consequences it is shown that an equationally compact ring satisfying both chain conditions is always finite, as is any Artinian ring which is a compact topological ring; further, using a result of S. Warner [11], we give a necessary and sufficient condition for an equationally compact Noetherian ring with identity to be a compact topological ring; a few remarks on the embedding of certain rings into equationally compact rings are made, and we obtain also here generalizations of known results on compact topological rings.

The material forms a part of the author's Ph.D. thesis.

Preliminary results. We begin by deriving a few useful tools. Let \( R \) be a ring and \( A \) an ideal of \( R \) ('ideal" always means two-sided ideal), and let \( \Sigma \) be a system of equations with constants in \( A \). If \( (x_{o}, x_{1}, \ldots, x_{y}, \ldots)_{y<\alpha} \) are the variables occurring in \( \Sigma \); then the solution set of \( \Sigma \) in \( R \) is a certain subset \( S \) of \( R^{\alpha} \). If such a system \( \Sigma \) exists such that the projection of \( S \) onto the first component is the ideal \( A \), then we shall say that \( A \) is \textit{expressible by equations}. For example, if \( R \) has an identity and \( A \) is finitely generated as a left ideal, then \( A \) is expressible by the equation \( x_{o} = x_{1}a_{1} + \ldots + x_{n}a_{n} \), where \( a_{1}, \ldots, a_{n} \) generate \( A \).

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If \( x \) is a variable and \( A \) is an ideal of \( R \), then "\( x \in A \)" will denote, quite naturally, the relational predicate \( A(x) \) for the unary relation \( A \) on \( R \).

We will make recurrent use of the following observation:

Remark. Let \( R \) be an equationally compact ring. Suppose \( (A_i \mid i \in I) \) is a family of ideals of \( R \), each of which is expressible by equations, and suppose \( (x_i \mid i \in I) \) is a family of variables. Let \( \Sigma \) be a set of equations with constants in \( R \). Then the system of formulas

\[
\Omega := \Sigma \cup \{x_i \in A_i \mid i \in I\}
\]

is solvable in \( R \) provided it is finitely solvable in \( R \).

proof: Let \( A_i \) be expressible by the system \( \Sigma_i, i \in I \); let \( (x_{0i}, x_{1i}, \ldots, x_{yi}, \ldots) \) denote the variables appearing in \( \Sigma_i \), whereby it is assumed that the variables \( x_{yi} \) and \( x_{\delta j} \) are distinct if \( i \neq j \) or \( \gamma \neq \delta \), and that no \( x_{yi} \) occurs in \( \Sigma \). Now the finite solvability of \( \Omega \) implies the finite solvability of the system of equations

\[
\bigcup (\Sigma_i \mid i \in I) \cup \{x_i = x_{0i} \mid i \in I\} \cup \Sigma,
\]

which is then solvable by the equational compactness of \( R \), and a solution obviously yields a solution of \( \Omega \) in \( R \). q.e.d.

Proposition 1. Let \( R \) be a ring and \( A \) an ideal of \( R \) such that \( A \) is expressible by equations and \( R \) is equationally compact. Then \( R/A \) and \( A \) are equationally compact rings.

proof: Suppose \( \Sigma = (\phi_i = 0 \mid i \in I) \) is a system of equations with constants in \( R/A \) and finitely solvable in \( R/A \). Now each \( \phi_i \) induces a polynomial in \( R \), say \( \phi'_i \), by replacing the constants by arbitrary representatives in \( R \). If \( z_i, i \in I \), are variables not occurring in \( \Sigma \), then the system

\[
\{\phi'_i = z_i \mid i \in I\} \cup \{z_i \in A \mid i \in I\}
\]

is clearly finitely solvable in \( R \), hence (by the last Remark) solvable in \( R \), and any solution taken modulo \( A \) yields a solution for \( \Sigma \) in \( R/A \). Thus \( R/A \) is equationally compact, and a similar argument shows that \( A \) is equationally compact.
Next we derive a useful remark on matrix rings.

**Proposition 2.** Let \( R \) be a ring with identity, let \( \mathfrak{m} \) be a nonzero cardinal and let \( S = M_{\mathfrak{m} \times \mathfrak{m}}(R) \) (i.e., \( S \) is the ring of linear transformations on the free \( R \)-module \( F \) on \( \mathfrak{m} \) generators). Then \( S \) is equationally compact if and only if \( R \) is equationally compact and \( \mathfrak{m} \) is finite.

**Proof:** Sufficiency. If \( \Sigma \) is a finitely solvable system of equations with constants in \( S \) then by replacing each variable \( x \) by the variable matrix \( (x_{ij})_{1 \leq i, j \leq \mathfrak{m}} \), every equation in \( \Sigma \) reduces in the obvious fashion to a system over \( R \), finitely solvable in \( R \), hence solvable in \( R \); such a solution yields a solution for \( \Sigma \) in \( S \).

Necessity. Let \( I \) be a set with cardinality \( \mathfrak{m} \) and let \( \{ e_i; i \in I \} \) be a basis for \( F \). Fix \( i_0 \in I \). For each \( i \in I \) define \( \pi_i \in S \) as follows: \( \pi_i(e_j) = \delta_{ij} e_{i_0} \) for all \( j \in I \). Let \( p_i \) be the retraction of \( F \) onto \( Re_i \). Then the system

\[
\Sigma = \{ p_i x = \pi_i; i \in I \}
\]

is finitely solvable (for a finite subset \( J \subseteq I \) of indices appearing, take \( x \) as follows: \( x(e_{i_0}) = \Sigma e_{i_0} \) and \( x(e_j) = 0 \) for \( j \neq i_0 \)). However \( \Sigma \) forces \( x \) to be such that \( x(e_{i_0}) = \Sigma e_{i_0} \) which is impossible unless \( \mathfrak{m} \) is finite. To see that \( R \) is equationally compact, consider a system \( \Sigma \) of equations with constants in \( R \) and finitely solvable in \( R \). For \( r \in R \) let \( e(r) \) denote the matrix \( (a_{ij}) \) where \( a_{11} = r \) and \( a_{ij} = 0 \) otherwise. Replace every constant \( r \in R \) appearing in \( \Sigma \) by \( e(r) \) and every variable \( x \) by \( e(1) \cdot x \cdot e(1) \). Then \( \Sigma \) is finitely solvable in \( S \), hence solvable in \( S \). Taking the upper left hand entries from a solution in \( S \) yields obviously a solution of \( \Sigma \) in \( R \). q.e.d.

If \( R = \langle R; +, -, 0, \cdot \rangle \) is a ring, we denote by \( R^+ \) the underlying additive abelian group \( \langle R; +, -, 0 \rangle \).

**Proposition 3.** Let \( R \) be an equationally compact ring and let \( D = \langle D; +, -, 0 \rangle \) be the largest divisible subgroup of \( R^+ \). Then \( R \cdot D = D \cdot R = \{ 0 \} \). In particular, \( D \) is an ideal of \( R \). Moreover, the ring \( R/D \) is equationally compact.

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proof: Let $d \in D$ and $r \in R$. Consider the system of equations

$$\Sigma = \{(x_i - x_j)x_{ij} = r \cdot d; i,j \in I, i \neq j\}$$

where $I$ is a set with cardinality larger than $|R|$. $\Sigma$ is finitely solvable in $R$, since for any finite subset of indices $J \subseteq I$, choose $n_i, i \in J$, to be distinct natural numbers, set $c_i = n_i r$, and pick $d_{ij}$ such that $(n_i - n_j)d_{ij} = d$ for $i \neq j$. Then clearly $(c_i - c_j)d_{ij} = r \cdot d$ for all $i,j \in J, i \neq j$. Thus $\Sigma$ must be solvable in $R$. However $\Sigma$ implies $x_i = x_j$ for some $i \neq j$, because of the cardinality of $I$, hence $r \cdot d = 0$. An almost identical argument shows that $d \cdot r = 0$.

We recall that an abelian group $G$ is algebraically compact (in the sense of Kaplansky [6]) if

$$G = \mathbb{C} \oplus (\prod G_p \mid p \text{ prime})$$

where $\mathbb{C}$ is divisible and each $G_p$ is a $p$-primary group complete in its $p$-adic topology and containing no nonzero element which is divisible by all powers of $p$. The group $R^+$ is equationally compact and therefore algebraically compact as was shown by S. Balcerzyk in [1]; thus in view of the latter condition on the $G_p$'s the subgroup $\mathcal{D}$ under discussion equals $\mathbb{C}$ and is expressible by the equations

$$\{x_0 = n \cdot x_n ; n \in \mathbb{N}\}.$$ 

Thus, $R/\mathcal{D}$ is equationally compact by Proposition 1.

Proposition 4. Let $R$ be an equationally compact ring such that $R^+$ is a bounded torsion group. Then there exists an equationally compact ring $S$ with identity such that $R$ is an ideal in $S$ of finite index.

proof: Let $n$ be a natural number such that $n \cdot R = (0)$, and let $\mathbb{Z}_n$ denote the integers modulo $n$. Define $S = <R \times \mathbb{Z}_n : +, -, 0, \cdot>$ as follows: $+$ is the usual direct sum addition, and

$$(r, l) \cdot (s, k) := (r \cdot s + l \cdot s + k \cdot r, l \cdot k).$$

The map $r \mapsto (r, 0)$ is a ring embedding of $R$ into $S$, making $R$ clearly an ideal of $S$ of finite index.

Now let $\Sigma$ be a system of equations with constants in $S$, finitely solvable in $S$. Let $(x_0, x_1, \ldots, x_\gamma, \ldots)_{\gamma < \alpha}$ be the variables appearing in $\Sigma$. Replace each variable $x_\gamma$ by $(y_\gamma, z_\gamma)$, inducing the system $\Sigma_0$ with the obvious interpretation of
solvability (i.e., $y_{\gamma}$ must be replaced by an element of $R$ and $z_{\gamma}$ by an element of $\mathbb{Z}_n^\alpha$). We construct by transfinite induction a sequence $(n_0, n_1, \ldots, n\gamma, \ldots)_{\gamma<\alpha} \in \mathbb{Z}_n^\alpha$, such that $\Sigma_0((z_{\gamma} \rightarrow n_{\gamma})_{\gamma<\alpha})$ is finitely solvable ("$z_{\gamma} \rightarrow n_{\gamma}$" means that the variable $z_{\gamma}$ is replaced by $n_{\gamma}$). Let $\beta$ be an ordinal and let $n_{\gamma}, \gamma<\beta$, be already constructed such that

$$\Sigma_\beta := \Sigma_0((z_{\gamma} \rightarrow n_{\gamma})_{\gamma<\alpha})$$

is finitely solvable. (For $\beta = 0$ the construction is trivial.) Suppose for each $m \in \mathbb{Z}_n$ the system $\Sigma_\beta(z_{\gamma} \rightarrow m)$ is not finitely solvable; i.e., for each $m \in \mathbb{Z}_n$ there exists a finite subset $\Sigma_{\beta,m}$ of $\Sigma_\beta$ such that $\Sigma_{\beta,m}(z_{\gamma} \rightarrow m)$ is not solvable. But then the finite system

$$\bigcup_{m \in \mathbb{Z}_n^\beta} \Sigma_{\beta,m} \leq \Sigma_\beta$$

is clearly not solvable. This is a contradiction, so there exists $n_{\beta} \in \mathbb{Z}_n$ such that $\Sigma_\beta(z_{\gamma} \rightarrow n_{\beta})$ is finitely solvable, and the induction step is complete. Thus $\Sigma_1 := \Sigma_0((z_{\gamma} \rightarrow n_{\gamma})_{\gamma<\alpha})$ is a finitely solvable system involving only the variables $(y_{\gamma})_{\gamma<\alpha}$.

Now any $\phi \in \Sigma_1$ is equivalent to a pair of equations $(\phi_1, \phi_2)$, where $\phi_1$ is an equation with constants in $R$ and involving the variables $(y_{\gamma})_{\gamma<\alpha}$, and $\phi_2$ involves only constants (from $\mathbb{Z}_n$). Therefore $\Sigma_1$ is solvable because $R$ is equationally compact. \textit{q.e.d.}

**Semisimplicity.** A ring $R$ is \textit{semisimple} if its Jacobson radical $J(R)$ is zero. We consider now the impact of this condition on equationally compact Artinian and Noetherian rings.

Recall that an element $r$ of a ring $R$ is \textit{left quasi-regular} if there exists an element $y \in R$ with $r + y + y \cdot r = 0$. It is well-known that $J(R)$ is the largest left quasi-regular left ideal in $R$; that is, $r \in J(R)$ if and only if the left ideal generated by $r$ is left quasi-regular. Hence $J(R)$ is expressible by the set of equations

$$(s \cdot x_0 + z \cdot x_0 + y_{s,z} + y_{s,z} \cdot (s \cdot x_0 + z \cdot x_0) = 0; s \in \mathbb{R}, z \in \mathbb{Z})$$

and in view of Proposition 1 we have

**Proposition 5.** If the ring $R$ is equationally compact, then so are the rings $R/J(R)$ and $J(R)$.
Lemma 1. A semisimple Artinian ring $R$ is equationally compact if and only if it is finite.

**proof:** Sufficiency. It is perhaps appropriate at this point to remark that an arbitrary universal algebra $A = \langle A; F \rangle$ which is also a compact topological algebra (i.e., $A$ can be endowed with a compact Hausdorff topology compatible with the algebraic structure) is equationally compact (see [8]). Indeed, the solution set of any equation is a closed subset of an appropriate power of $A$ endowed with the Tychonov product topology.

As a special case, any finite algebra, hence any finite ring, is equationally compact.

Necessity. It is easily seen that a finite direct sum of rings is equationally compact if and only if every summand is. By Wedderburn's theorem $R$ is a finite direct sum of matrix rings over division rings, each of which, therefore, is equationally compact. By Proposition 2 the respective divisions rings are equationally compact. However, equationally compact division rings are known to be finite (consider, for example, the system $E = \{(x_i - x_j)y_{ij} = 1; i,j \in I, i \neq j\}$ for suitably large $I$). Thus $R$ is finite.

**Proposition 6.** Let $R$ be an equationally compact semisimple Noetherian ring with identity. Then $R$ is finite.

In view of the fact that equationally compact Noetherian rings with identity are necessarily linearly compact for the discrete topology, Proposition 6 follows from D. Zelinsky's decomposition of linearly compact semisimple rings [15, Prop. 11] and Lemma 1. For completeness' sake we give a proof, which is in the spirit of an argument of S. Warner [12, p. 55].

Lemma 2. Let $R$ be as above but, in addition, a primitive ring. Then $R$ is finite (and hence simple Artinian).

**proof:** By the Jacobson-Chevalley Density Theorem $R$ is a dense ring of linear transformations on a vector space $V$ with basis, say, $\{e_i; i \in I\}$. For each $i \in I$, let

$$A_i = \{\phi \in R; \phi(e_i) = 0\}.$$  

$A_i$ is a left ideal, hence finitely generated, and therefore...
expressible by equations. Let \((v_i)_{i \in I} \in V^I\) be chosen arbitrarily. By denseness there exists for each \(i \in I\) \(\phi_i \in R\) such that \(\phi_i(e_i) = v_i\). Thus the system of equations

\[
\Sigma = \{x = \phi_i + z_i; i \in I\} \cup \{z_i \in A_i; i \in I\}
\]

is finitely solvable (again by denseness) and hence solvable. However \(\Sigma\) implies that \(x\) must map each \(e_i\) to \(v_i\). Thus \(R\) is the complete transformation ring, and therefore by Prop. 2 and Lemma 1 a finite matrix ring over a division ring. \(\text{q.e.d.}\)

Proof of Proposition 6: As is well-known \(R\) is a subdirect product of a family of primitive rings \(\{R/A_i; i \in I\}\) where the \(A_i\)'s are ideals of \(R\). Since \(R\) is Noetherian with identity, each \(A_i\) is expressible by equations, so \(R/A_i\) is equationally compact by Proposition 1 and Noetherian. Hence by Lemma 2 \(R/A_i\) is finite, simple and Artinian. Hence the \(A_i\)'s are maximal ideals. Let \(r = (r_i + A_i)_{i \in I} \in \Pi(R/A_i; i \in I)\). The system

\[
\Sigma = \{x = r_i + z_i; i \in I\} \cup \{z_i \in A_i; i \in I\}
\]

is finitely solvable by the Chinese Remainder Theorem, hence solvable in \(R\). But \(\Sigma\) implies \(x = r_i\), so \(r \in R\). Hence \(R\) is the full direct product and so \(I\) must be finite because \(R\) is Noetherian. \(\text{q.e.d.}\)

We summarize these results in the following

**Theorem 1.** For an equationally compact, semisimple ring \(R\) the following are equivalent:

(i) \(R\) is finite.

(ii) \(R\) is Artinian.

(iii) \(R\) is Noetherian with identity.

**Noetherian rings.** Although we are not able to characterize structurally those Noetherian rings with identity which are equationally compact, Theorem 1 and a crucial result of Warner yield a pleasant criterion relating equational compactness and topological compactness in this class of rings. We paraphrase the relevant result:
Proposition 7 [11, Theorem 2]. Let $R$ be a topological Noetherian ring with identity. Then $R$ is topologically compact if and only if the topology of $R$ is the radical topology $T$, $R$ is complete for that topology and $R/J(R)$ is a finite ring.

Now let $R$ be an equationally compact Noetherian ring with identity. By Theorem 1, $R/J(R)$ is finite. Now the topology $T$ defined by taking the powers of $J(R)$ as a neighbourhood base of 0 is not necessarily Hausdorff. However, we shall show that the space $(R, T)$ is complete. To see this, consider a Cauchy sequence $(r_i)_{i=1,2,\ldots}$ in $R$. For each natural number $n$ choose $i_n$ such that the subsequence $(r_{i_n})_{i=1,2,\ldots}$ is $J(R)^n$-close. Since $R$ is Noetherian with identity, the ideal $J(R)^n$ is expressible by equations, so we have the system of equations

$$
\Sigma = \{x = r_i + z_n; n \in \mathbb{N}\} \cup \{z_n \in J(R)^n; n \in \mathbb{N}\}
$$

which is finitely solvable (if $m$ is the largest index appearing in a finite subset, set $x = r_i$ and $z_n = r_i - r_i$ for all $n \geq m$). Hence $\Sigma$ is solvable and obviously any solution is a limit of $(r_i)_{i=1,2,\ldots}$. As a matter of fact, $T$ is compact. To see this we quote the following Lemma 3. Let $R$ be a ring with identity, $A$ and $B$ two ideals such that $B$ is finitely generated as a left ideal and both $R/A$ and $R/B$ are finite. Then $R/A \cdot B$ is finite.

The proof is a straightforward counting of cosets as given in the proof of [10, Lemma 4], where the hypothesized commutativity is not used.

Now by Lemma 3 and induction, we see that $J(R)^n$ has finite index in $R$ for each $n$. This means that the family of cosets $F = \{r + J(R)^n; r \in R, n \in \mathbb{N}\}$ is a subbase of closed sets for the topology $T$, and by the Alexander Subbase Theorem $T$ is compact if every subfamily of $F$ with the finite intersection property has a nonempty intersection. The latter is however clear by equational compactness of $R$ and the fact that each $J(R)^n$ is expressible by equations. In view of Proposition 7 we have proved
Theorem 2. Let $R$ be an equationally compact Noetherian ring with identity. Then the radical topology is a complete and compact topology on $R$, and $R/J(R)$ is finite. Moreover, $R$ is a compact topological ring if and only if $\bigcap(J(R)^n | n \in \mathbb{N}) = \{0\}$.

Remark. By [3], equational and topological compactness coincide when $R$ is a commutative Noetherian ring with identity. In general, I do not know of an equationally compact Noetherian ring with identity which is not topologically compact.

Artinian rings. As an immediate consequence of Theorem 2 we have the following

Corollary 1. An equationally compact Artinian ring $R$ with identity is finite.

proof: Two well-known results assert that $R$ is Noetherian and $J(R)$ is nilpotent. Hence $J(R)^n = (0)$ for some $n$, thus the radical topology is discrete and, by theorem 2, compact, which forces $R$ to be finite.

Corollary 2 [11, Theorem 2, Corollary]. A compact topological Artinian ring with identity is finite.

The case of arbitrary Artinian rings requires a closer look.

Lemma 4. If $R$ is an equationally compact Artinian ring such that $R^+$ is a bounded torsion group, then $R$ is finite.

proof: By Proposition 4 there is an equationally compact ring with identity $S$, such that $R$ is an ideal of $S$ and $S/R$ is finite. Thus $R$ is an Artinian $S$-module, as is the finite $S$-module $S/R$, and so $S$ is an Artinian $S$-module, i.e., $S$ is an Artinian ring. But then $S$ is finite by Corollary 1. q.e.d.

Lemma 5. Let $R$ be an equationally compact torsion-free Artinian ring. Then $R = (0)$.

proof: A torsion-free Artinian ring has, as well-known, a left identity $e$ and is an algebra over the rationals. But then the system of equations
\[(x_i - x_j)Y_{ij} = e; \ i,j \in I, \ i \neq j\]
is finitely solvable in \(R\), hence solvable in \(R\); taking \(|I| > |R|\)
forces \(e = 0\), i.e., \(R = (0)\).

Recall that the Prüfer group \(\mathbb{Z}(p^\infty)\) is the subgroup of
the unit circle in the complex plane consisting of all \(p^n\)-th
roots of unity for all natural numbers \(n\) and fixed prime \(p\).

**Theorem 3.** For an Artinian ring \(R\) the following are equivalent:

(i) \(R\) is equationally compact.

(ii) \(R^+ = B \oplus P\) where \(B = \langle B; +, \cdot, 0 \rangle\) is a finite group,

\(P = \langle P; +, \cdot, 0 \rangle\) is a finite direct sum of Prüfer
groups, and \(R \cdot P = P \cdot R = \{0\}\).

(iii) \(R\) is (algebraic) retract of a compact topological

**proof:** (iii) \(\Rightarrow\) (i) holds for arbitrary universal algebras
(see [8]).

(i) \(\Rightarrow\) (ii): By a result of F. Szász [9, Satz 4] every Artinian
ring is the ring direct sum of its torsion ideal \(T\) and some
torsion-free ideal \(D\). But \(D\) is then an equationally compact
torsion-free Artinian ring, so must be \((0)\) by Lemma 5. Hence
\(R = T\). Let \(R^+ = B \oplus P\) be the (group) decomposition of \(R^+\) into
its divisible part \(P\) and reduced part \(B\). As a torsion divisible
abelian group \(P\) is, as well-known, a direct sum of Prüfer
groups. Now by Proposition 3 \(R \cdot P = P \cdot R = \{0\}\). Thus every
subgroup of \(P\) is an ideal of \(R\) and therefore \(P\) is a finite
direct sum, because \(R\) is Artinian.

Now the family \(F = \{n \cdot B \oplus P; \ n \in \mathbb{N}\}\) is easily seen to be a
downward directed set of ideals of \(R\), hence has a smallest
element \(n_0 \cdot B \oplus P\) since \(R\) is Artinian. However \(n_0 \cdot B \oplus P\) is
clearly divisible, being the meet of \(F\), and so \(n_0 \cdot B = (0)\)
as \(B\) is reduced. Thus \(B\) is a bounded torsion group. The
quotient \(R/P\) is Artinian and, again by Proposition 3, equationally compact; moreover, \((R/P)^+ = B\). Hence \(B\) is finite by
Lemma 4, and we are done.

(ii) \(\Rightarrow\) (iii): Let \(R^+ = B \oplus P_1 \oplus \cdots \oplus P_n\) where \(B\) is finite
and \(P_i = \mathbb{Z}(p_i^\infty)\), \(i = 1, \ldots, n\). Each \(P_i\) is divisible, hence injective and therefore retract of every extending abelian group —
e.g., the compact topological circle group \(C\). Let \(f_i : C \to P_i\)
be a retraction. Endowing $B$ with the discrete topology, we have then a (group) retraction

$$f: H \to R^+$$

where $H$ is the compact topological group $B \oplus (\oplus(C_i \mid i=1,\ldots,n))$ and $f = \text{id}_B \oplus f_1 \oplus \cdots \oplus f_n$.

If multiplication is defined on $H$ by letting every element of $\oplus(C_i \mid i=1,\ldots,n)$ annihilate $H$ and then extending by distributivity, $H$ clearly becomes a ring. Moreover $H$ is a topological ring under the given topology, because the inverse image under the multiplication map of any subset of $H$ is the finite union of sets of the form $A_1 \times A_2$ where each $A_j$ is a coset of $\oplus(C_i \mid i=1,\ldots,n)$ in $H$, all of which, however, are closed; thus multiplication is continuous. By a straightforward calculation one sees that $f$ is a ring homomorphism, and the proof is complete.

Remark. It is not possible, in general, to obtain a ring-direct sum in the decomposition given in condition (ii). Consider, for example, the ring $R$, where $R^+ = Z_2 \oplus Z(2^\infty)$, $R \cdot Z(2^\infty) = Z(2^\infty) \cdot R = \{0\}$, and $(1,0) \cdot (1,0)$ is defined to be the primitive square root of unity in $Z(2^\infty)$. Here we have a nonzero divisible element appearing as a product of two nondivisible elements.

The following improves Corollary 2:

**Corollary 3.** A compact topological Artinian ring $R$ is finite.

**proof:** By Theorem 3 we have $R^+ = B \oplus P_1 \oplus \cdots \oplus P_n$, where $B$ is finite and $P_i = Z(p_i^{\infty})$. Let $P_i^k$ be the subgroup of $P_i$ consisting of all $p_i^k$-th roots of unity, and let

$$R^k = B \oplus P_1^k \oplus \cdots \oplus P_n^k.$$ 

Now $R = \bigcup(R^k \mid k=1,2,3,\ldots)$, that is, the intersection of the complements $R \setminus R^k$ is empty. By the Baire Category Theorem [7, p.200] at least one of the sets $R \setminus R^k$ is not dense in $R$, i.e., for some $k_0$ the finite subgroup $R^{k_0}$ contains a nonempty open set; this forces the topology to be discrete and therefore by compactness $R$ must be finite.

**Corollary 4.** An equationally compact ring satisfying both chain conditions is finite.

**proof:** clear.
Compactifications. We conclude with a few remarks on the question of embedding rings into equationally compact ones. Following the terminology of [14] we define, for a fixed universal algebra $A$, a compactification of $A$ to be an algebra $B$ such that $B$ is equationally compact and $A$ is a subalgebra of $B$. $B$ is a quasi-compactification of $A$ if $A$ is a subalgebra of $B$ and every system of equations with constants in $A$ and finitely solvable in $A$ is solvable in $B$. The classes of compactifications resp. quasi-compactifications of $A$ are denoted by $\text{Comp}(A)$ resp. $\text{c}(A)$. Clearly $\text{Comp}(A) \subseteq \text{c}(A)$. A positive formula is a formula of the first order predicate calculus which is built up from polynomial equations (of a fixed algebraic type) by application of the logical connectives $\lor$, $\exists$, $\land$, $\lor$ in a finite number of steps. We quote the following result of G.H. Wenzel:

**Proposition 8** [14, Theorems 8.10,12]. Let $A$ be an algebra and let $K$ be one of $\text{Comp}(A)$ or $\text{c}(A)$. If $K$ is not empty then there is an algebra $B$ in $K$ such that $B$ satisfies every positive formula with constants in $A$ which is satisfiable in $A$.

**Proposition 9.** Let $R$ be a ring and $\Lambda$ an infinite division ring. If $R$ contains $\Lambda$ as a subring, then $c(R) = \emptyset$. In particular, an infinite semisimple Artinian ring cannot be quasi-compactified, and hence not (algebraically) embedded into a compact topological ring. If $R$ is an algebra over $\Lambda$ and $R^2 \neq \{0\}$, then $c(R) = \emptyset$. If $D$ denotes any divisible subgroup of $R^+$ and $R \cdot D \neq \{0\}$, then $c(R) = \emptyset$. In particular, if $R$ is a subring of a compact topological ring, then $R \cdot D = D \cdot R = \{0\}$.

**Proof:** If $c(R) \neq \emptyset$, then $c(R)$ contains a ring by Proposition 8; the proofs are then implicit in Proposition 3.

**Proposition 10.** Let $R$ be an infinite Artinian ring with identity. Then $\text{Comp}(R) = \emptyset$. In particular, $R$ cannot be (algebraically) embedded in a compact topological ring.

**Proof:** $R$ is Noetherian by a well-known result; hence $R$ has finite length. If $n$ is the (unique!) length of a maximal chain of left ideals then as is easily checked, the property
of "maximal length of at most n" is characterized by the positive formula

$$\varphi = (\forall x_1) \cdots (\forall x_{n+2})(\exists y_1) \cdots (\exists y_{n+2})$$

$$\bigvee_{1<k<n+2} x_k = y_1 x_1 + \cdots + y_{k-1} x_{k-1}$$

Thus if \( \text{Comp}(R) \neq \emptyset \), there is by Proposition 8 an \( S \in \text{Comp}(R) \) satisfying \( \varphi \), i.e., of finite length. But this cannot be, since by Corollary 4 \( S \) would be finite.

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