On the Eigenvectors of a Finite-Difference Approximation to the Sturm-Liouville Eigenvalue Problem

by

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1. Introduction

The present contribution is concerned with the nonselfadjoint problem

\[-[a(x)u_x]_x - b(x)u_x + c(x)u = \lambda u, \quad 0 < x < 1,\]

(1)

\[u(0) = u(1) = 0\]

where \(a(x) \geq a > 0\), \(c(x) \geq 0\), and \(a, b, c\) are all bounded and smooth functions. This problem has an infinite sequence of positive and distinct eigenvalues

\[0 < \lambda_1 < \lambda_2 < \lambda_3 < ...\]

and a corresponding sequence of smooth eigenfunctions \(u^1, u^2, u^3, \ldots\) (see for instance Protter-Weinberger [10, p. 37] and Coddington-Levinson [4, p. 212]). Following Courant-Hilbert [5, p. 334] the eigenfunctions \(u^p\) are uniformly bounded in the supremum norm if they are normalized so that

\[\int_0^1 |u^p(x)|^2 dx = 1, \quad p = 1, 2, 3, \ldots\]

Of course, by the well-known transformation

\[u(x) = \exp\left(-\frac{1}{2} \int_0^x \frac{b(t)}{a(t)} dt\right)w(x)\]

(2) may be put in the selfadjoint form

\[-[a(x)w_x]_x + \hat{c}(x)w = \lambda w, \quad 0 < x < 1,\]

\[w(0) = w(1) = 0\]
where
\[ \hat{c}(x) = c(x) + \frac{1}{2} b_x(x) + \frac{1}{4} b^2(x)/a(x). \]
Here, in order to obtain \( \hat{c}(x) \geq 0 \) we have to make a restricting assumption on \( b_x \). Therefore we choose the direct approximation of (1) by means of the finite-difference equations
\[
\begin{align*}
-a_{k+1/2}(v_{k+1} - v_k) - & a_{k-1/2}(v_k - v_{k-1}) \quad \frac{\Delta x^2}{2b_k} v_{k+1} - v_{k-1} \\
+ c_k v_k &= \Lambda v_k, \\
\end{align*}
\]
(3)

where \( M \in \mathbb{N}, \Delta x = 1/(M+1), \) and \( v_k = v(k\Delta x) \). Equivalently, we may write (3) in matrix-vector notation
\[
(3') \quad LV = \Lambda V
\]
where \( V = (v_1, \ldots, v_M)^T \) and the matrix \( L \) may be easily derived from (3).

Let \( |b(x)| \leq \beta \) and \( 0 \leq \Delta x \leq 2\alpha/\beta \). Then the matrix \( L \) is equivalent to a real symmetric matrix (see Carasso [2]). Using this fact and Theorem 1.8 of Varga [11] it can be shown that all eigenvalues \( \Lambda_p \) of (3) are real and positive,
\[
0 \leq \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots \leq \Lambda_M,
\]
and there exists a complete sequence of corresponding eigenvectors \( V^p \). A result of Carasso [2, Corollary 1] says that there exist a constant \( K \) and an integer \( p_0 \), both independent of \( M \), such that
\[ |\mathbf{v}^p|_\infty = \max_{1 \leq k \leq M} |\mathbf{v}^p_k| \leq K \rho^{1/2}, \quad \rho_0 \leq \rho \leq M, \]

if

\[ |\mathbf{v}^p|^2 = \Delta x \sum_{k=1}^{M} |\mathbf{v}^p_k|^2 = 1. \]

In the selfadjoint case this result goes back to Büchner [1]. In this paper we prove the following theorem:

**Theorem.** Let \( a(x) \geq \alpha > 0 \) and \( c(x) \geq 0, \ 0 \leq x \leq 1 \). Assume that \( a, b, \) and \( c \) are differentiable bounded functions with bounded derivatives; say \( |b(x)| \leq \beta \). Let 
\( 0 \leq \Delta x \leq \alpha / \beta \) and let \( (\mathbf{v}^p)^M_{p=1} \) be the eigenvectors of (3) normalized so that \( |\mathbf{v}^p|^2 = 1 \). Then

\[ |\mathbf{v}^p|_\infty \leq \kappa, \quad p = 1, \ldots, M, \]

for some constant \( \kappa \) independent of \( M \).

**Remark 1.** In the case of the equation \( u_{xx} = \lambda u \) this result may be proved by explicit computation of the eigenvectors \( \mathbf{v}^p \) (see Isaacson-Keller [9, 9.1.1]).

Applications of the Theorem to the theory of finite-difference approximations to parabolic and hyperbolic partial differential equations are given in [6, 7].
2. Proof of the Theorem

Instead of $L$ we consider, as in [2], the eigenvectors $D^{-1}V^P$ of the similar matrix $D^{-1}LD$ defined below. But in contrast to Carasso [2], who uses a discrete maximum principle for his estimation, we transpose then the proof of Courant-Hilbert [5, p. 334] to the resulting discrete problem.

The following basic results are needed.

Lemma 1 (Carasso [2, Lemma 1, 3, Lemma 3.1]). Let $D = (d_1, \ldots, d_M)$ be the diagonal matrix with

\[ d_1 = 1, \quad d_i = \left[ \frac{i-1}{\sum_{k=1}^{i-1} a_{k+1/2} - b_{k+1} \Delta x/2} \right]^{1/2}, \quad i = 2, \ldots, M. \]

For $0 \leq \Delta x \leq 2a/\beta$ we have $d_i > 0$ and

\[ |D|_{\infty} \leq \kappa_1, \quad |D^{-1}|_{\infty} \leq \kappa_2 \]

for constants $\kappa_1, \kappa_2$ independent of $M$. Furthermore

\[ D^{-1}LD = (P + Q)/\Delta x^2 \]

where $P = (p_{ik})_{i,k=1,\ldots,M}$

\[ p_{ik} = \begin{cases} 
(p_{k+1/2} + p_{k-1/2}) & \text{if } i = k \\
-p_{k+1/2} & \text{if } i = k+1 \\
-p_{i+1/2} & \text{if } k = i+1 \\
0 & \text{otherwise}
\end{cases} \]

\[ p_{k+1/2} = (a_{k+1/2} - b_{k+1} \Delta x/2)^{1/2}(a_{k+1/2} + b_{k} \Delta x/2)^{1/2}, \]

and $Q = (q_1, \ldots, q_M)$ is the diagonal matrix with

\[ q_k = (a_{k+1/2} + a_{k-1/2}) - (p_{k+1/2} + p_{k-1/2}) + \Delta x^2 c_k. \]
Remark 2. The change of variables $V = DW$ is a discrete analog to (2) [2].

Lemma 2 (Carasso [2, Theorem 1]). Let $\lambda_p, v^p$ be the characteristic pairs of the matrix $L$ with $|v^p|_2 = 1$. Let $u^p$ be an eigenfunction of (1) corresponding to $\lambda_p$ and let $U^p$ be the vector of dimension $M$ obtained from $u^p$ by mesh-point evaluation. Assume $u^p$ normalized so that $|D^{-1}u^p|_2 = |D^{-1}v^p|_2$ then as $\Delta x \to 0$, we have

$$|\lambda_p - \lambda_p| \leq \kappa_3(p)\Delta x^2,$$

$$|U^p - V^p|_2 \leq \kappa_4(p)\Delta x^2$$

where $\kappa_3, \kappa_4$ are positive constants depending only on $p$.

In the selfadjoint case Lemma 2 was proved by Gary [6].

Remark 3. The estimation (4) implies

$$|u^p - v^p|_\infty \leq \kappa_4(p)\Delta x^{3/2}.$$

Lemma 3. Let

$$C_1(W) = \sum_{k=1}^1 \left[ p_{k+1/2}(w_k - w_{k+1}) - p_{k-1/2}(w_k - w_{k-1}) \right] q_k w_k / \Delta x^2.$$

Then, under the assumptions of the Theorem,

$$C_1(w^p) = -q_1 p_{1+1/2} w_{p+1}^p w_1^p + \sigma(1), \quad l = 1, \ldots, M,$$

where $\sigma(1)$ denotes a function which has a bound independent of $M$.

Proof. We show at first that $|q_k/\Delta x^2| \leq \kappa_5$ independently of $M$.

To this end it suffices to consider $a_{k+1/2} = p_{k+1/2} - p_{k+1/2}$. 

By means of the binomial theorem we obtain
\[
a_{k+1/2} - p_{k+1/2} = a_{k+1/2} - a_{k+1/2} \left(1 - \frac{b_{k+1} \Delta x}{4a_{k+1/2}}\right) + \sigma(\Delta x^2) \left(1 + \frac{b_k \Delta x}{4a_{k+1/2}}\right).
\]

Inserting \( b_{k+1} = b_k + \sigma(\Delta x) \) we find that
\[
(5) \quad a_{k+1/2} - p_{k+1/2} = \sigma(\Delta x^2).
\]

Now, since \( w_o^p = 0 \),
\[
C_1(w^p) = -q_1 p_{1+1/2} w_{1+1}^p w_1^p / \Delta x^2
\]
\[
\sum_{k=1}^{l-1} \frac{q_k}{\Delta x^2} (p_{k+1/2} - p_{k-1/2}) w_k^p w_k^p + \sum_{k=1}^{l-1} \frac{q_{k+1} - q_k}{\Delta x^2} p_{k+1/2} w_{k+1}^p w_k^p.
\]

But by the mean value theorem we have
\[
P_{k+1} - p_k = \sigma(\Delta x) \text{ and } (q_{k+1} - q_k) / \Delta x^2 = \sigma(\Delta x). \text{ Hence,}
\]

using Schwarz's inequality and \( |w^p|^2 \leq \kappa_6 \) we obtain the desired result.

Now according to Lemma 1 it suffices to prove the Theorem for the eigenvectors \( w^p = D^{-1}v^p \) of the matrix \((P+Q) / \Delta x^2\) which has the eigenvalues \( \Lambda_p \) too. We multiply the \( k \)-th row of
\[
\frac{1}{\Delta x^2}(P + Q)w^p = \Lambda_p w^p
\]
by
\[
p_{k+1/2}(w_k^p - w_{k+1}^p) - p_{k-1/2}(w_k^p - w_{k-1}^p)
\]
and obtain by adding all rows from \( k = 1 \) until \( k = l \)
\[
\left[ \frac{p_{1+1/2}(w_1^p - w_{1+1}^p)}{\Delta x} \right]^2 + C_1(w^p)
\]
\[
+ \lambda_p p_{1+1/2} w_{1+1}^p w_1^p - \lambda_p \Delta x \sum_{k=1}^{l-1} \frac{1}{p_x} (\xi_k) w_k^p w_k^p = \left[ \frac{p_{1/2}(w_1^p - w_o^p)}{\Delta x} \right]^2.
\]
where \((k-1/2)\Delta x < \xi_k < (k+1/2)\Delta x\). In order to eliminate the term on the right side of (6) we sum up the equations (6) for \(l = 1\) to \(l = M\), add \((p_{1/2}^P(w_1^P - w_o^P)/\Delta x)^2\) to both sides, and divide by \(M+1 = 1/\Delta x\). Then

\[
\left[\frac{p_{1/2}^P(w_1^P - w_o^P)}{\Delta x}\right]^2 = \Delta x \sum_{l=0}^{M} \left[\frac{p_{l+1/2}^P(w_{l+1}^P - w_l^P)}{\Delta x}\right]^2 + \Delta x \sum_{l=1}^{M} C_l(w^P) \cdot (7)
\]

But

\[
0 \leq \alpha/2 \leq p_{l+1/2}^P \leq \kappa_7
\]

if \(0 \leq \Delta x \leq \alpha/\beta\). Thus, using the fundamental relation

\[
\sum_{l=0}^{M} p_{l+1/2}(w_{l+1} - w_l)^2 = W_{PW}^T
\]

and Schwarz's inequality we derive

\[
\Delta x \sum_{l=0}^{M} \left[\frac{p_{l+1/2}^P(w_{l+1}^P - w_l^P)}{\Delta x}\right]^2 + \Delta x \kappa_7 \sum_{l=1}^{M} \frac{q_1^P w_{l+1}^P}{\Delta x} - \Delta x \kappa_7 \sum_{l=1}^{M} \frac{q_{1}^P w_{l}^P}{\Delta x} \\
\leq \kappa_7 \lambda_p + \kappa_5 \kappa_7
\]

since \(|q_{1}/\Delta x^2| \leq \kappa_5\) independently of \(M\). Hence, applying Schwarz's inequality once more we find from (7) by means of the Assumption and Lemma 3 that

\[
\left[\frac{p_{1/2}^P(w_1^P - w_o^P)}{\Delta x}\right]^2 \leq \kappa_8 \lambda_p + \kappa_9.
\]

From this estimation, equation (6), and Lemma 3 we deduce that

\[
(\lambda_p - \frac{q_{1}}{\Delta x^2})p_{l+1/2}^P w_{l+1}^P w_l^P \leq \kappa_{10} \lambda_p + \kappa_{11}.
\]
Consequently, observing (8) we obtain, in case \( \Lambda_p > \kappa_5 \), that

\[
\omega_{1+1}^p \omega_1^p \leq \kappa_{12}^1, \quad l = 1, \ldots, M,
\]

for some constant \( \kappa_{12} \) independent of \( M \). For \( \Lambda_p \leq \kappa_5 \) the assertion of the Theorem follows by Lemma 2.

Finally, we return once more to equation (6). The above estimations yield

\[
\left[ \frac{p_{l+1/2}(\omega_{1+1}^p - \omega_1^p)}{\Delta x} \right]^2 \leq \kappa_{13} \Lambda_p + \kappa_{14}, \quad l = 1, \ldots, M,
\]

or, using (9),

\[
\max_{1 \leq l \leq M} \{ \omega_{1+1}^p, \omega_1^p \} \leq \Delta x^2 (\kappa_{15} \Lambda_p + \kappa_{16}) + \kappa_{17} \leq \kappa
\]

because \( \Delta x^2 \Lambda_p \) is bounded independently of \( M \).

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References


