Common Extensions of Positive Vector Measures

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We prove a general theorem on the existence of a positive common extension of a family of positive vector measures in an order complete Riesz space. Our theorem gives an easy access to vector-valued versions of results due to Guy, Horn and Tarski, and Marczewski.

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1. Introduction

The following result was proposed by Guy [3]:

1.1. Proposition.

Let \( \Omega \) be a set, let \( M \) and \( N \) be algebras of subsets of \( \Omega \), and let \( \mu : M \to \mathbb{R} \) and \( \nu : N \to \mathbb{R} \) be positive additive set functions. Then the following are equivalent:

(a) \( \mu(A) \leq \nu(B) \) and \( \nu(C) \leq \mu(D) \) holds for all \( A, D \in M \) and \( B, C \in N \) satisfying \( A \subseteq B \) and \( C \subseteq D \).

(b) There exists a positive additive set function \( \varphi : 2^\Omega \to \mathbb{R} \) satisfying \( \varphi(A) = \mu(A) \) for all \( A \in M \) and \( \varphi(A) = \nu(A) \) for all \( A \in N \).

This result is remarkable since it characterizes the existence of a positive common extension in terms of the set functions alone; by contrast, no such result is known in the case of bounded additive set functions, where some condition on the algebras seems to be indispensable; see Lipecki [8] and Schmidt and Waldschaks [12]. Moreover, Proposition 1.1 cannot be extended to more than two set functions; see Bhaskara Rao and Bhaskara Rao [2; Example 3.6.3].

Unfortunately, the proof of Proposition 1.1 given by Guy [3] is incorrect, as will be made precise in Section 4 of this paper, and the proofs given by Bhaskara Rao and Bhaskara Rao [2; Theorem 3.6.1] and Kindler [5,6] are rather extensive. In the present paper we prove a general extension theorem for families of positive vector measures which gives an easy access to vector-valued versions of Proposition 1.1 and results due to Horn and Tarski [4] and Marczewski [10,11].
Throughout this paper, let $\Omega$ be a set and let $\mathcal{G}$ be an order complete Riesz space. Let us first recall some definitions and facts which will be needed in the sequel:

For a Riesz space $\mathcal{H}$, a linear operator $T : \mathcal{H} \to \mathcal{G}$ is positive if $Tx \in \mathcal{G}_+$ holds for all $x \in \mathcal{H}_+$. For further details on Riesz spaces and linear operators, see [1].

For an algebra $\mathcal{F}$ of subsets of $\Omega$, a vector measure $\phi : \mathcal{F} \to \mathcal{G}$ is positive if it maps $\mathcal{F}$ into $\mathcal{G}_+$. Let $\mathcal{E}(\mathcal{F}) := \text{lin} \{ \chi_A \mid A \in \mathcal{F} \}$ and define $\chi : \mathcal{F} \to \mathcal{E}(\mathcal{F})$ by letting

$$\chi(A) := \chi_A,$$

where $\chi_A$ denotes the indicator function of $A \in \mathcal{F}$. Then $\mathcal{E}(\mathcal{F})$ is a Riesz space with order unit $\chi_\Omega$, and $\chi$ is a positive vector measure. Moreover, each vector measure $\phi : \mathcal{F} \to \mathcal{G}$ defines its representing linear operator $T : \mathcal{E}(\mathcal{F}) \to \mathcal{G}$, given by

$$T \left( \sum_{i=1}^{n} a_i \chi_{A_i} \right) := \sum_{i=1}^{n} a_i \phi(A_i),$$

and each linear operator $T : \mathcal{E}(\mathcal{F}) \to \mathcal{G}$ defines a vector measure $\phi : \mathcal{F} \to \mathcal{G}$, given by

$$\phi := T \circ \chi.$$

Obviously, $\phi$ is positive if and only if $T$ is positive.
2. Positive operators

The following extension theorem is a consequence of the Hahn-Banach theorem for linear operators; for a proof, see [1; Theorem 2.8]:

2.1. Proposition.
Let $E$ be a Riesz space with order unit $e \in E_+$, let $F$ be a subspace of $E$ satisfying $e \in F$, and let $S : F \to G$ be a positive operator. Then there exists a positive operator $T : E \to G$ satisfying $Tx = Sx$ for all $x \in F$.

For a Riesz space $E$ and a family $\{ E_\delta \mid \delta \in \Delta \}$ of subspaces of $E$, let $\phi( E_\delta \mid \delta \in \Delta )$ denote the collection of all families $\{ x_\delta \in E_\delta \mid \delta \in \Delta \}$ satisfying $x_\delta \neq 0$ for at most finitely many $\delta \in \Delta$. A family $\{ T_\delta : E_\delta \to G \mid \delta \in \Delta \}$ of linear operators has a common extension if there exists a linear operator $T : E \to G$ satisfying $Tx = T_\delta x$ for all $\delta \in \Delta$ and $x \in E_\delta$.

2.2. Theorem.
Let $E$ be a Riesz space with order unit $e \in E_+$ and let $\{ E_\delta \mid \delta \in \Delta \}$ be a family of subspaces of $E$ satisfying $e \in \bigcap_\Delta E_\delta$. For a family $\{ T_\delta : E_\delta \to G \mid \delta \in \Delta \}$ of positive operators, the following are equivalent:

(a) $\sum_\Delta T_\delta x_\delta \in G_+$ holds for each family $\{ x_\delta \} \in \phi( E_\delta \mid \delta \in \Delta )$ satisfying $\sum_\Delta x_\delta \in E_+$.

(b) The family $\{ T_\delta \}$ has a positive common extension $T : E \to G$. 
Proof. It is sufficient to prove that (a) implies (b). Define $F := \text{lin} \left( \bigcup_{\Delta} E_{\delta} \right)$. Then the mapping $S : F \to \mathbb{C}$, given by

$$Sx := \Sigma_{\Delta} T_{\delta} x_{\delta}$$

for all $x \in F$ and arbitrary $\{x_{\delta}\} \in \mathcal{F}(E_{\delta} | \delta \in \Delta)$ satisfying $x = \Sigma_{\Delta} x_{\delta}$, is well-defined and linear, and it is also positive. Now the assertion follows from Proposition 2.1.

Theorem 2.2 is due to Maharam [9].
3. Positive vector measures

For a family \( \{ F_\delta \mid \delta \in \Delta \} \) of algebras of subsets of \( \Omega \), a family \( \{ \Phi_\delta : F_\delta \to \mathcal{G} \mid \delta \in \Delta \} \) of vector measures has a common extension if there exists a vector measure \( \Phi : 2^\Omega \to \mathcal{G} \) satisfying \( \Phi(A) = \Phi_\delta(A) \) for all \( \delta \in \Delta \) and \( A \in F_\delta \).

3.1. Theorem.

Let \( \{ F_\delta \mid \delta \in \Delta \} \) be a family of algebras of subsets of \( \Omega \). For a family \( \{ \Phi_\delta : F_\delta \to \mathcal{G} \mid \delta \in \Delta \} \) of positive vector measures, the following are equivalent:

(a) \[ \sum_{i=1}^{m} \Phi_\delta(i)(A_i) \leq \sum_{i=m+1}^{m+n} \Phi_\delta(i)(A_i) \] holds for all \( m, n \in \mathbb{N} \), all \( A_1, \ldots, A_{m+n} \in \bigcup_\Delta F_\delta \) satisfying
\[ \sum_{i=1}^{m} \chi_{A_i} \leq \sum_{i=m+1}^{m+n} \chi_{A_i} \], and all \( \delta(1), \ldots, \delta(m+n) \in \Delta \) satisfying \( A_i \in F_\delta(i) \) for all \( i \in \{1, \ldots, m+n\} \).

(b) The family \( \{ \Phi_\delta \} \) has a positive common extension \( \Phi : 2^\Omega \to \mathcal{G} \).

Proof. It is sufficient to prove that (a) implies (b). For all \( \delta \in \Delta \), define \( E_\delta := E(F_\delta) \) and let \( T_\delta : E_\delta \to \mathcal{G} \) denote the representing linear operator of \( \Phi_\delta \). We claim that
\[ 0 \leq \sum_\Delta T_\delta g_\delta \]
holds for each family \( \{ g_\delta \} \in \Phi( E_\delta \mid \delta \in \Delta ) \) satisfying \( 0 \leq \sum_\Delta g_\delta \). Indeed, this is obvious for families of simple functions taking their values in \( \mathbb{Z} \), by the assumption on \( \{ \Phi_\delta \} \), and hence for families of simple functions taking their values in \( \mathbb{Q} \). Consider now an arbitrary family \( \{ g_\delta \} \in \Phi( E_\delta \mid \delta \in \Delta ) \) satisfying \( 0 \leq \sum_\Delta g_\delta \), and let \( m \) denote the number of \( \delta \in \Delta \) for which \( g_\delta \neq 0 \).

For each \( k \in \mathbb{N} \) and \( \delta \in \Delta \) choose \( g_{\delta,k} \in E_\delta \) such that
each \( g_{\delta,k} \) takes its values in \( \mathbb{Q} \) and satisfies \( g_{\delta,k} = 0 \) if \( g_{\delta} = 0 \) and
\[
g_{\delta,k} - \frac{1}{km} \chi_Q \leq g_{\delta} \leq g_{\delta,k}.
\]
Then we have, for all \( k \in \mathbb{N} \),
\[
0 \leq \Sigma_{\Delta} g_{\delta,k}
\]
and hence
\[
0 \leq \Sigma_{\Delta} T_\delta g_{\delta,k} \leq \Sigma_{\Delta} T_\delta g_{\delta} + \frac{1}{k} T_\chi_Q.
\]
Since \( \mathcal{G} \) is order complete and hence Archimedean, we obtain
\[
0 \leq \Sigma_{\Delta} T_\delta g_{\delta},
\]
which proves our claim. Define now \( \mathbb{E} := \mathbb{E}(2^\Omega) \). By what we have shown and Theorem 2.2, the family \( \{T_\delta\} \) has a positive common extension \( T : \mathbb{E} \rightarrow \mathcal{G} \), and it is then clear that the vector measure \( \varphi : 2^\Omega \rightarrow \mathcal{G} \), given by
\[
\varphi := T \chi_X,
\]
is a positive common extension of the family \( \{\varphi_\delta\} \).

In the case \( \mathcal{G} = \mathbb{R} \), Theorem 3.1 is equivalent to a result of Lembcke [7].

As a consequence of Theorem 3.1 we obtain the following vector-valued version of Proposition 1.1:

3.2. Corollary.
Let \( F_1 \) and \( F_2 \) be algebras of subsets of \( \Omega \). For positive vector measures \( \varphi_1 : F_1 \rightarrow \mathcal{G} \) and \( \varphi_2 : F_2 \rightarrow \mathcal{G} \), the following are equivalent:

(a) \( \varphi_i(A_i) \leq \varphi_j(A_j) \) holds for all \( i,j \in \{1,2\} \) and all \( A_i \in F_1 \) and \( A_j \in F_j \) satisfying \( A_i \subseteq A_j \).

(b) \( \varphi_1 \) and \( \varphi_2 \) have a positive common extension \( \varphi : 2^\Omega \rightarrow \mathcal{G} \).
Proof. Consider \( m, n \in \mathbb{N} \) and \( A_1, \ldots, A_{m+n} \in F_1 \cup F_2 \) satisfying
\[
\sum_{i=1}^{m} x_{A_i} \leq \sum_{i=m+1}^{m+n} x_{A_i}
\]
and hence
\[
\sum_{i=1}^{m+n} x_{A_i} \leq n x_\Omega.
\]
For \( i \in \{1, \ldots, m+n\} \), define
\[
C_i := \begin{cases} A_i & \text{if } A_i \in F_1 \\ \emptyset & \text{otherwise} \end{cases}
\]
and
\[
D_i := A_i \setminus C_i.
\]
Define now
\[
g := \sum_{i=1}^{m+n} x_{C_i} \quad \text{and} \quad h := \sum_{i=1}^{m+n} x_{D_i}.
\]
Then we have \( g \in \mathcal{E}(F_1) \) and \( h \in \mathcal{E}(F_2) \), and \( g + h \leq n x_\Omega \).

For \( k \in \{1, \ldots, n\} \), define
\[
M_k := \{ \omega \in \Omega \mid k x_\Omega(\omega) \leq g(\omega) \}
\]
and
\[
N_k := \{ \omega \in \Omega \mid k x_\Omega(\omega) \leq n x_\Omega(\omega) - h(\omega) \}
\]
Then we have \( M_k \in F_1 \) and \( N_k \in F_2 \), and we also have \( M_k \subseteq N_k \)
and thus
\[
\varphi_1(M_k) \leq \varphi_2(N_k).
\]
Using
\[
\sum_{i=1}^{m+n} x_{C_i} = g = \sum_{k=1}^{n} x_{M_k}
\]
and
\[
\sum_{k=1}^{n} x_{N_k} = n x_\Omega - h = n x_\Omega - \sum_{i=1}^{m+n} x_{D_i},
\]
and the previous inequality, we obtain
\[ \sum_{i=1}^{m+n} \varphi_1(C_i) = \sum_{k=1}^{n} \varphi_1(M_k) \]
\[ \leq \sum_{k=1}^{n} \varphi_2(N_k) \]
\[ = n\varphi_2(\Omega) - \sum_{i=1}^{m+n} \varphi_2(D_i) , \]
hence
\[ \sum_{i=1}^{m+n} (\varphi_1(C_i) + \varphi_2(D_i)) \leq n\varphi_2(\Omega) , \]
whence
\[ \sum_{i=1}^{m+n} \varphi_j(i)(A_i) \leq n\varphi_2(\Omega) = n\varphi_1(\Omega) , \]
and thus
\[ \sum_{i=1}^{m} \varphi_j(i)(A_i) \leq \sum_{i=m+1}^{m+n} \varphi_j(i)(\overline{A_i}) , \]
for all \( j(1), \ldots, j(m+n) \in \{1,2\} \) satisfying \( A_i \in F_j(i) \) for all \( i \in \{1,\ldots,m+n\} \). The assertion now follows from Theorem 3.1.

We now record two further applications of Theorem 3.1:

**3.3. Corollary.**

Let \( C \) be a collection of subsets of \( \Omega \) satisfying \( \emptyset, \Omega \in C \).

If \( \zeta : C \to \mathcal{G} \) is a set function such that
\[ \sum_{i=1}^{m} \zeta(C_i) \leq \sum_{i=m+1}^{m+n} \zeta(C_i) \]
holds for all \( m, n \in \mathbb{N} \) and \( C_1, \ldots, C_{m+n} \in C \) satisfying
\[ \sum_{i=1}^{m} \chi_{C_i} \leq \sum_{i=m+1}^{m+n} \chi_{C_i} , \]
then there exists a positive vector measure \( \varphi : 2^\Omega \to \mathcal{G} \)
satisfying \( \varphi(C) = \zeta(C) \) for all \( C \in C \).
Proof. For each \( C \in \mathcal{C} \), let \( F_C \) denote the algebra generated by \( C \) and define a positive vector measure \( \varphi_C : F_C \rightarrow \mathcal{G} \) by letting \( \varphi_C(C) := \zeta(C) \) and \( \varphi_C(\overline{C}) := \zeta(\Omega) - \zeta(C) \); note that the assumption on \( \zeta \) yields \( \varphi_C(\overline{C}) = \zeta(\overline{C}) \) for all \( C \in \mathcal{C} \). Consider \( m, n \in \mathbb{N} \) and \( A_1, \ldots, A_{m+n} \in \bigcup_C F_C \) satisfying
\[
\sum_{i=1}^{m} x_{A_i} \leq \sum_{i=m+1}^{m+n} x_{A_i}
\]
and thus
\[
\sum_{i=1}^{m+n} x_{A_i} \leq n x_{\Omega}.
\]
Relabelling the \( A_i \) if necessary, we obtain
\[
A_1, \ldots, A_p, \overline{A}_{p+1}, \ldots, \overline{A}_{m+n} \in \mathcal{C}
\]
for some \( p \in \{0,1,\ldots,m+n\} \). Then we have
\[
\sum_{i=1}^{p} x_{A_i} + (m+n-p) x_{\Omega} \leq \sum_{i=p+1}^{m+n} x_{\overline{A}_i} + n x_{\Omega},
\]
hence
\[
\sum_{i=1}^{m+n} x_{\zeta(A_i)} + (m+n-p) x_{\zeta(\Omega)} \leq \sum_{i=p+1}^{m+n} x_{\zeta(\overline{A}_i)} + n x_{\zeta(\Omega)},
\]
whence
\[
\sum_{i=1}^{m+n} \varphi_C(i)(A_i) \leq n \zeta(\Omega),
\]
and thus
\[
\sum_{i=1}^{m} \varphi_C(i)(A_i) \leq \sum_{i=m+1}^{m+n} \varphi_C(i)(\overline{A}_i),
\]
for all \( C(1), \ldots, C(m+n) \in \mathcal{C} \) satisfying \( A_i \in F_C(i) \) for all \( i \in \{1,\ldots,m+n\} \). The assertion now follows from Theorem 3.1.

In the case \( \mathcal{G} = \mathbb{R} \), Corollary 3.3 is due to Horn and Tarski [4]; see also Lembcke [7].
3.4. Corollary.

Let $\mathcal{C}$ be a collection of subsets of $\Omega$ such that

$$( \bigcap_{D} D ) \cap ( \bigcap_{E} E ) \neq \emptyset$$

holds for any two disjoint finite subcollections $D$ and $E$ of $\mathcal{C}$.

If $\zeta : \mathcal{C} \rightarrow \mathbb{G}$ is a set function which maps $\mathcal{C}$ into an order bounded subset of $\mathbb{G}^+$, then there exists a positive vector measure $\varphi : 2^\Omega \rightarrow \mathbb{G}$ satisfying $\varphi(C) = \zeta(C)$ for all $C \in \mathcal{C}$.

Proof. Let $\mu := \sup_{C} \zeta(C)$. For each $C \in \mathcal{C}$, let $\mathcal{F}_C$ denote the algebra generated by $C$ and define a positive vector measure $\varphi_C : \mathcal{F}_C \rightarrow \mathbb{G}$ by letting

$$\varphi_C(C) := \zeta(C) \quad \text{and} \quad \varphi_C(\emptyset) := \mu - \zeta(C).$$

Consider $m, n \in \mathbb{N}$ and $A_1, \ldots, A_{m+n} \in \bigcup_{C} \mathcal{F}_C$ satisfying

$$\sum_{i=1}^{m} x_{A_i} \leq \sum_{i=m+1}^{m+n} x_{A_i}$$

and thus

$$\sum_{i=1}^{m+n} x_{A_i} \leq nx_{\Omega}.$$ 

We now reduce the previous inequality by subtracting

$$x_{A_i} = 0 \quad \text{if} \quad A_i = \emptyset,$$

$$x_{A_i} = x_{\Omega} \quad \text{if} \quad A_i = \Omega,$$

$$x_{A_i} + x_{A_j} = x_{\Omega} \quad \text{if} \quad A_i \text{ and } A_j \text{ are complementary.}$$

Relabelling the $A_i$ if necessary, we thus obtain

$$\sum_{i=1}^{p+q} x_{A_i} \leq kx_{\Omega},$$

for suitable $p, q, k \in \mathbb{N} \cup \{0\}$ satisfying $p+q \leq m+n$ and $k \leq n$ as well as

$$A_1, \ldots, A_p, \bar{A}_{p+1}, \ldots, \bar{A}_{p+q} \in \mathcal{C}.$$
By the assumption on $C$, we have
\[
\left( \bigcap_{i=1}^{p} A_i \right) \cap \left( \bigcap_{i=p+1}^{p+q} A_i \right) \neq \emptyset.
\]
This yields $p+q \leq k$, hence
\[
\sum_{i=1}^{p+q} \varphi_C(i)(A_i) \leq (p+q) \varphi_C(\Omega) \leq ku,
\]
whence, reversing the previous reduction,
\[
\sum_{i=1}^{m+n} \varphi_C(i)(A_i) \leq nu,
\]
and thus
\[
\sum_{i=1}^{m} \varphi_C(i)(A_i) \leq \sum_{i=m+1}^{m+n} \varphi_C(i)(\overline{A}_i),
\]
for all $C(1), \ldots, C(m+n) \in C$ satisfying $A_i \in F_C(i)$ for all $i \in \{1, \ldots, m+n\}$. The assertion now follows from Theorem 3.1.

In the case $G = \mathbb{R}$, Corollary 3.4 is due to Marczewski [10, 11]; see also Lembcke [7].
4. **Remark**

The following example shows that the proof of Proposition 1.1 given by Guy [3] is incorrect:

In the notation of [3], define $X := [0, 1)$ as well as

$F_1 := [0, \frac{1}{2})$, $F_2 := [\frac{1}{2}, 1)$, $G_1 := [0, \frac{1}{3})$, $G_2 := [\frac{1}{3}, \frac{2}{3})$, $G_3 := (\frac{2}{3}, 1]$, let $R$ and $S$ denote the algebras generated by the sets $F_1$, $F_2$ and $G_1$, $G_2$, $G_3$, respectively, let $\lambda : R \rightarrow \mathbb{R}$ and $\mu : S \rightarrow \mathbb{R}$ denote the restrictions of the Lebesgue measure to these algebras, and define $N := 2$ and $M := 3$ as well as $a_1 := -1$, $a_2 := -1$, $b_1 := 1$, $b_2 := 1$, $b_3 := 1$. With these definitions, the final equality in the formula following (12) is false, and this is also true for the inequality by which it may be replaced.
References


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