Daugavet's Equation and Orthomorphisms

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Abstract. The main result of this paper asserts that each Dunford-Pettis operator on an AL-space having no discrete elements satisfies Daugavet's equation \( \| I + T \| = 1 + \| T \| \); this extends a result of Holub on weakly compact operators. The proof is based on some properties of orthomorphisms in a Banach lattice, which also yield a short proof of another result of Holub on Daugavet's equation for bounded operators on an arbitrary AL- or AM-space.

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1. Introduction

A linear operator $T : E \rightarrow E$ on a Banach space $E$ satisfies Daugavet's equation if

$$|| I + T || = 1 + || T ||$$

holds, where $I : E \rightarrow E$ denotes the identity operator. Daugavet's equation clearly fails for $T := -I$, but it holds under suitable conditions on $E$ and $T$.

The first results on Daugavet's equation were obtained by Daugavet [6] and Lozanovskii [13] who proved that the identity

$$|| I + T || = 1 + || T ||$$

holds for each compact operator on $C[0,1]$ or $L^1[0,1]$. These results were subsequently extended into various directions [4,5,7-12,16]; in particular, it follows from results of Foias and Singer [8] and Holub [9,10] that Daugavet's equation remains valid for each weakly compact operator on $C[0,1]$ or $L^1[0,1]$, and that each bounded operator on these spaces satisfies at least one of the identities $|| I + T || = 1 + || T ||$ and $|| I - T || = 1 + || T ||$.

In the present paper we shall study Daugavet's equation for linear operators on a Banach lattice. Using some properties of orthmorphisms, we shall prove that Daugavet's equation holds for each Dunford-Pettis operator on an AL-space having no discrete elements, and that each bounded operator on an arbitrary AL- or AM-space satisfies at least one of the identities $|| I + T || = 1 + || T ||$ and $|| I - T || = 1 + || T ||$. The first of these results extends a result of Holub [10] on weakly compact operators; the second is essentially due to Holub [9,10] and has recently been given a short proof by Abramovich [1], but the proof given here is equally short and avoids the use of representation theorems.
Throughout this paper, let $\mathcal{E}$ be a Banach lattice, let $L(\mathcal{E})$ denote the normed ordered vector space of all bounded operators $\mathcal{E} \to \mathcal{E}$, and let $I : \mathcal{E} \to \mathcal{E}$ denote the identity operator. A linear operator $Q : \mathcal{E} \to \mathcal{E}$ is an orthomorphism if it is order bounded and if $Q(B) \subseteq B$ holds for each band $B$ of $\mathcal{E}$. Let $\text{Orth}(\mathcal{E})$ denote the Riesz space [3; Theorem 8.9] of all orthomorphisms $\mathcal{E} \to \mathcal{E}$. If $\mathcal{E}$ is either an AL-space or an order complete AM-space with unit, then $L(\mathcal{E})$ is an order complete Banach lattice [3; Theorem 15.3 and the remark preceding it] and $\text{Orth}(\mathcal{E})$ agrees with the (projection) band generated by $I$ in $L(\mathcal{E})$ [3; Theorem 8.11]. This property of $\text{Orth}(\mathcal{E})$ together with Lemma 2.3 below indicates a natural connection between Daugavet's equation and orthomorphisms on Banach lattices.
2. The results

We start with a simple but useful lemma on positive operators:

2.1. Lemma. Let $E$ be an AL- or AM-space. Then Daugavet's equation holds for each positive $T \in \mathcal{L}(E)$.

Proof. Suppose first that $E$ is an AL-space and consider a positive operator $T : E \to E$. Then

$$
\| (I+T)z \| = \| z \| + \| Tz \|
$$

holds for each $z \in E^+$, and this yields

$$
\| I + T \| = 1 + \| T \| .
$$

In the case where $E$ is an AM-space, the assertion now follows by duality.\[\Box\]

Our next result concerns bounded operators which are not necessarily positive:

2.2. Theorem. Let $E$ be an AL- or AM-space. Then the identity

$$
\max \{ \| I+T \| , \| I-T \| \} = 1 + \| T \|
$$

holds for each $T \in \mathcal{L}(E)$.

Proof. Let us first assume that $E$ is an order complete AM-space with unit $e \in E^+_+$.

For each $U \in \text{Crel}(E)$, we have

$$
\| I+I-U \| = I + \| U \|
$$

and thus

$$
(1) \quad \max \{ \| I+U \| , \| I-U \| \} = \| I+U \| \| I-U \|
\quad = \| I+U \| \| U \|
\quad = 1 + \| U \|.\]
by [3; Theorem 15.5] and Lemma 2.1.

Consider now $T \in L(E)$ and choose $S \in \text{Orth}(E)$ and $R \in \text{Orth}(E)^\perp$ satisfying

$$T = S + R.$$  

Since $|R|e$ is dominated by a scalar multiple of $e$, there exists a positive $Q \in \text{Orth}(E)$ satisfying

$$Qe = |R|e,$$

by [3; Theorem 8.15]. Moreover, for each $P \in \text{Orth}(E)$, we have

$$|P + Q|v|P - Q| = |P| + Q$$

and

$$|P| + |R| = |P + R| = |P - R|,$$

hence

$$(|P + Q|v|P - Q|)e = |P + R|e = |P - R|e,$$

and thus

$$(2) \quad \max \{ \|P + Q\|, \|P - Q\| \} = \|P + R\| = \|P - R\|.$$  

Replacing $P$ by $S$, $I + S$, and $I - S$ in $(2)$, we obtain

$$\max \{ \|S + Q\|, \|S - Q\| \} = \|T\|$$

$$\max \{ \|I + S + Q\|, \|I + S - Q\| \} = \|I + T\|$$

$$\max \{ \|I - S + Q\|, \|I - S - Q\| \} = \|I - T\|;$$

similarly, replacing $U$ by $S + Q$ and $S - Q$ in $(1)$, we obtain

$$\max \{ \|I + S + Q\|, \|I - S - Q\| \} = 1 + \|S + Q\|$$

$$\max \{ \|I + S - Q\|, \|I - S + Q\| \} = 1 + \|S - Q\|.$$  

This yields

$$\max \{ \|I + T\|, \|I - T\| \}
= \max \{ \|I + S + Q\|, \|I + S - Q\|, \|I - S + Q\|, \|I - S - Q\| \}
= 1 + \|T\|.$$  

In the case where $E$ is an AL-space or an arbitrary AM-space, the assertion now follows by duality.  

The following result is another consequence of Lemma 2.1:

2.3. **Lemma.** Let $E$ be an AL-space of an order complete AM-space with unit. Then Daugavet's equation holds for each $T \in L(E)$ satisfying $|I \land T| = 0$.

**Proof.** By assumption, we have

$$|I + T| = I + |T|$$

and thus

$$||I + T|| = ||I + |T|| = 1 + ||T||,$$

by Lemma 2.1.

We now turn to the main result of this paper. Recall that a linear operator $E \rightarrow E$ is a Dunford-Pettis operator if it maps the weakly convergent sequences of $E$ into the norm convergent sequences of $E$, and that every Dunford-Pettis operator is bounded. Let $\mathcal{D}(E)$ denote the subspace of $L(E)$ consisting of all Dunford-Pettis operators $E \rightarrow E$. Also, recall that an element $u \in E \setminus \{0\}$ is discrete if the ideal generated by $u$ in $E$ agrees with the subspace generated by $u$ in $E$.

2.4. **Theorem.** Let $E$ be an AL-space having no discrete elements. Then Daugavet's equation holds for each $T \in \mathcal{D}(E)$.

**Proof.** Consider $T \in \mathcal{D}(E)$ and define $S := |I \land T|$. Since $\text{Orth}(E)$ as well as $\mathcal{D}(E)$ are bands of $L(E)$ [3; Theorems 15.5 and 19.15], we have $S \in \text{Orth}(E) \cap \mathcal{D}(E)$. 
Consider now \( z \in \mathcal{E}_+ \). For each \( y \in [0, Sz] \), there exists some \( Q \in \text{Orth}(\mathcal{E}) \) satisfying
\[
y = QSz
\]
and \( Qz \in [0, z] \), by [3; Theorem 8.15 and its proof]. Since \( S \) is an orthomorphism, we obtain
\[
y = QSz = SQz \in S[0, z],
\]
by [3; Theorems 8.24 and 8.21]. This yields
\[
[0, Sz] = S[0, z].
\]
Since \( S \) is also a Dunford-Pettis operator, the set \( S[0, z] \) is compact [3; Theorem 19.18]. Thus, the order interval \( [0, Sz] \) is compact, hence \( Sz \) belongs to the band generated by the collection of all discrete elements of \( \mathcal{E} \) [2; Theorem 21.12], and the assumption on \( \mathcal{E} \) yields \( Sz = 0 \).

Therefore, we have \( \|T\| = S = 0 \), and the assertion follows from Lemma 2.3.

Under the assumption of Theorem 2.4, every weakly compact operator on \( \mathcal{E} \) is a Dunford-Pettis operator, but the converse is not true; see [3; Theorems 19.6 and 19.23].

We finally note that Theorem 2.4 cannot be extended to arbitrary AL-spaces: In fact, if \( \mathcal{E} \) is an AL-space having a discrete element \( u \in \mathcal{E}_+ \backslash \{0\} \), then the band \( B\{u\} \) generated by \( u \) is a one-dimensional subspace of \( \mathcal{E} \), by [14; Proposition 8.3]; consequently, the band projection \( P : \mathcal{E} \to B\{u\} \) is compact, but Daugavet's equation fails for \( T := -P \).
3. Remarks.

The following results can be proven in the same way as Lemmas 2.1 and 2.3:

3.1. Lemma. Let $\mathcal{E}$ be a Banach lattice satisfying
\[ \| x \|^p + \| y \|^p = \| x + y \|^p \]
for some $p \in [1, \infty)$ and all $x, y \in \mathcal{E}_+$. If $J \in \mathcal{L}(\mathcal{E})$ is a positive isometry, the
\[ \| J + T \| = 1 + \| T \| \]
holds for each positive $T \in \mathcal{L}(\mathcal{E})$.

3.2. Lemma. Let $\mathcal{E}$ be an AL-space. If $J \in \mathcal{L}(\mathcal{E})$ is a positive isometry, then
\[ \| J + T \| = 1 + \| T \| \]
holds for each $T \in \mathcal{L}(\mathcal{E})$ satisfying $Ja|T| = 0$.

Corresponding results hold in an AM-space or an order complete AM-space with unit, respectively, if in Lemmas 2.1 and 2.3 the identity operator is replaced by a surjective positive isometry.

Obviously, Daugavet's equation holds for $T := I$, and this implies that the condition of Lemma 2.3 is only sufficient, but not necessary, for Daugavet's equation to hold. Also, the fact that Daugavet's equation fails for $T := -I$ can be generalized as follows: If $T \in \mathcal{L}(\mathcal{E})$ satisfies $0 < T < I$, then $\| I - T \|$ is strictly smaller than $1 + \| T \|$.

We conclude with a brief discussion of Daugavet's equation for almost integral and absolute kernel operators.
Let $E$ be an AL-space or an order complete AM-space with unit. A linear operator $T : E \rightarrow E$ is an almost integral operator if it is contained in the band of $L(E)$ which is generated by the linear operators $S : E \rightarrow E$ satisfying $Sz = x'(z)y$ for some $x' \in E$ and $y \in E$ depending on $S$ and all $z \in E$.

Synnatzschke [15] proved that $E$ has no discrete elements if and only if each almost integral operator $T : E \rightarrow E$ satisfies $\int |T| = 0$, and in this case it follows from Lemma 2.3 that $T$ satisfies Daugavet's equation. In the case where $E$ is an AL-space, this result can also be deduced from [17; Theorem 123.5(ii)], [3; Theorem 19.18], and Theorem 2.4; in the case where $E$ is an order complete AM-space with unit, it is due to Synnatzschke [16].

Consider now a $\sigma$-finite measure space $(\Omega, F, \mu)$ and $p \in [1, \infty)$. A linear operator $T : L^p(\Omega, F, \mu) \rightarrow L^p(\Omega, F, \mu)$ is an absolute kernel operator if there exists an $F \times F$-measurable function $t : \Omega \times \Omega \rightarrow \mathbb{R}$ satisfying

$$\int_{\Omega} |t(., \omega)| f(\omega) \, d\mu(\omega) \in L^p(\Omega, F, \mu)$$

and

$$(Tf)(.) = \int_{\Omega} t(., \omega)f(\omega) \, d\mu(\omega)$$

for all $f \in L^p(\Omega, F, \mu)$. If $(\Omega, F, \mu)$ has no atoms, then $L^p(\Omega, F, \mu)$ has no discrete elements and a linear operator $L^p(\Omega, F, \mu) \rightarrow L^p(\Omega, F, \mu)$ is an absolute kernel operator if and only if it is an almost integral operator [17; Theorem 94.7]. Thus, for $p \in [1, \infty)$, Daugavet's equation holds for each absolute kernel operator $L^p(\Omega, F, \mu) \rightarrow L^p(\Omega, F, \mu)$ whenever $(\Omega, F, \mu)$ has no atoms.
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