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Optimal Sovereign Default

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Optimal Sovereign Default[*]

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Abstract

When is it optimal for a government to default on its legal repayment obligations? We answer this question for a small open economy with domestic production risk in which the government optimally finances itself by issuing non-contingent debt. We show that Ramsey optimal policies occasionally deviate from the legal repayment obligation and repay debt only partially, even if such deviations give rise to significant 'default costs'. Optimal default improves the international diversification of domestic output risk, increases the efficiency of domestic investment and - for a wide range of default costs - significantly increases welfare relative to a situation where default is simply ruled out from Ramsey optimal plans. We show analytically that default is optimal following adverse shocks to domestic output, especially for very negative international wealth positions. A quantitative analysis reveals that for empirically plausible wealth levels, default is optimal only in response to disaster-like shocks to domestic output, and that default can be Ramsey optimal even if the net foreign asset position is positive.

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1 Introduction

When is it optimal for a sovereign to default on its outstanding debt? We analyze this hotly debated question in a quantitative equilibrium framework in which a country can borrow and invest internationally to smooth out the consumption implications

of domestic productivity shocks. Importantly, we determine the default policies that maximize the country's *ex ante* welfare, i.e. derive the Ramsey optimal policy under full commitment. We show that Ramsey policies involve sovereign default, i.e. it is optimal for the government to occasionally repay less than is legally required to serve its outstanding debt contracts. This is optimal even if default events give rise to sizeable deadweight costs. A quantitative analysis suggests that optimal default policies significantly increase *ex ante* welfare, relative to a situation where sovereign default is simply ruled out by assumption.

The fact that sizeable welfare gains can arise from sovereign default may appear surprising, given that policy discussions and also the academic literature tend to emphasize the inefficiencies associated with sovereign default events. Popular discussions, for example, tend to focus on the potential *ex post* costs associated with a sovereign default, such as the adverse consequences for the functioning of the banking sector or the economy as a whole. While certainly relevant, we show that sovereign default can remain optimal, even if default costs of an empirically plausible magnitude arise. Likewise, following the seminal contribution of Eaton and Gersovitz (1981), the academic literature also tends to emphasize the inefficiencies created by default decisions: anticipation of default in the future limits the ability to issue debt today, thereby constraining the ability to smooth out adverse shocks.

Our analysis emphasizes that sovereign default also fulfills a very useful economic function, even in a setting with a fully committed government: a default engineers a resource transfer from lenders to the sovereign debtor in times when resources are scarce on the sovereign's side. The option to default thus provides insurance against adverse economic developments in domestic income. This point has previously been emphasized by Grossmann and Van Huyck (1988), who coined the term 'excusable default' to capture default events that are the result of an implicit risk sharing agreement between the sovereign borrower and its lenders. Assuming the absence of default costs, Grossman and Van Huyck (1988) study whether the optimal allocation with cost-free 'excusable default' can be sustained as a reputational equilibrium in a setting without a committed borrower. The present analysis abstracts entirely from issues related to lack of commitment, and is instead concerned with characterizing the optimal allocation with 'excusable default', but for the empirically more plausible setting with non-zero default costs. We do so using a production economy with a general shock structure, which generalizes the setup with iid endowment shocks analyzed in Grossman and Van Huyck' seminal work. And as we prove analytically, the presence of default costs strongly affects optimal allocations and the optimal default policy, with the default policy being discontinuously affected when moving from zero to positive default cost levels.

Sovereign default is Ramsey optimal in our setting because government bond markets are incomplete, such that bond markets do not provide any explicit insurance against domestic income shocks.[1] The incompleteness of government bond markets thereby emerges *endogenously* from the presence of contracting frictions, which we

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[1] Moreover, adjustments of the domestic investment margin only partially contribute to smoothing domestic consumption.

describe in detail in section 3 of the paper. These frictions make it optimal for the government to issue debt contracts which - in legal terms - promise a repayment amount that is not contingent on future events. This is in line with empirical evidence, which shows that existing government debt consists predominantly of non-contingent debt instruments.[2] The contracting framework represents an important advance over earlier work studying Ramsey optimal government policy under commitment and incomplete markets, which simply assumes the incompleteness of government bond markets (e.g. Sims (2001), Angeletos (2002), Ayiagari et al. (2002), or Adam (2011)). Importantly, the contracting framework also provides microfoundations for the presence of 'default costs'.

Using this setting with non-contingent sovereign debt and default costs, we extend the existing Ramsey policy literature by treating repayment of debt as a (continuous) decision variable in the optimal policy problem. We show analytically that, for a wide range of empirically plausible default cost specifications, the assumption of full debt repayment is inconsistent with fully optimal behavior. While full repayment is optimal if the country has accumulated a sufficient amount of international wealth, which then serves as a buffer against adverse domestic shocks, full repayment is suboptimal for sufficiently low wealth levels, provided default costs do not take on prohibitive values.[3] The presence of non-zero default costs is key for this finding, as with cost-free default the optimal default patterns are independent of the country's wealth position.

Besides providing analytical characterizations of the optimal default policies, we also quantitatively assess the economic conditions under which sovereign default is part of Ramsey optimal policy. For this purpose, we provide a lower bound estimate for the costs of default implied by our structural model and use it as an input for our quantitative analysis. We show that plausible levels of default costs make it optimal for the government *not* to default following business cycle sized shocks to productivity, thereby vindicating the full repayment assumption often entertained in the Ramsey policy literature with incomplete markets. Only when the country's net foreign debt position approaches its maximum sustainable level, does sovereign default become optimal following an adverse business cycle shock.

Given that reasonably sized default costs largely eliminate sovereign default in response to business cycle sized shocks, we introduce economic 'disaster' risk into the aggregate productivity process, following Barro and Jin (2011). Default then reemerges as part of optimal government policy, following the occurrence of a sufficiently severe disaster shock. This is the case even for sizeable default costs and even when the country's net foreign asset position is far from its maximally sustainable

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[2]Most sovereign debt is non-contingent only in nominal terms, and could be made contingent by adjusting the price level, a point emphasized by Chari, Kehoe and Christiano (1991). As shown in Schmitt-Grohe and Uribe (2004), however, such price level adjustments are suboptimal in the presence of even modest nominal rigidities. Morevoer, for countries that are members of a monetary union, non-contingent nominal debt is effectively non-contingent in real terms, since the country cannot control the price level.

[3]Default costs are prohibitive if the costs of default are equal to, or higher than the amount of resources not repaid to lenders.

level. Indeed, for a severe output disaster it can be optimal to default even if the country's net foreign asset position is positive. For the U.S. net foreign asset position, which stood at -26,7% of GDP at the end of 2011, our model suggests an optimal default amount of around 3.16 trn U.S. dollars, should the U.S. economy experience an economic disaster in which output falls approximately 30% below trend. For smaller output disasters in the order of 8% of GDP, default remains suboptimal, and this continues to be true for considerably more negative net foreign asset positions.

We also investigate the welfare consequences of using government default by comparing the optimal policy with default to a situation where the government is *assumed* to repay debt unconditionally. In the latter setting, adjustments in the international wealth position and of the domestic investment margin are the only channels available for smoothing domestic consumption. The consumption equivalent welfare gains associated with optimal default decisions easily reach 1 percentage point of consumption each period and prove surprisingly robust to the level of default costs.

In related work, Sims (2001) discusses fiscal insurance in the context of whether or not Mexico should dollarize its economy. Considering a setting where the government is assumed to issue only non-contingent nominal debt that is assumed always to be repaid, he shows how giving up the domestic currency allows for less insurance, as it deprives the government of the possibility of using price adjustments to alter the real value of outstanding debt. The present paper considers a model with real bonds that are optimally non-contingent and allows for outright government debt default. Our setting could thus be reinterpreted as one where bonds are effectively non-contingent in nominal terms, but where the country has delegated the control of the price level to a monetary authority that pursues price stability, say by dollarizing or by joining a monetary union. As we then show, in such a setting the default option still provides the country with a possible and quantitatively relevant insurance mechanism.

Angeletos (2002) explores fiscal insurance in a closed economy setting with incomplete government bond markets, assuming full repayment of debt. He shows how a government can use the maturity structure of domestic government bonds to insure against domestic shocks, by exploiting the fact that bond yields of different maturities react differently to shocks. This channel is unavailable in our small open economy setting, since the international yield curve does not react to domestic events.

The remainder of the paper is structured as follows. Section 2 introduces the economic environment, formulates the Ramsey policy problem, and derives the necessary and sufficient conditions characterizing optimal policy. In doing so, we also introduce a new approach for proving concavity of Ramsey problems involving default decisions and show how to properly impose 'natural borrowing limits' in quantitative model applications. Both issues should be of interest in a variety of other applications. To simplify the exposition, we assume in this section that the government can issue non-contingent debt only and that deviations from the legally stated repayment promise gives rise to proportional default costs. Section 3 then endogenizes the government debt contract and derives the optimality of non-contingent government debt and the presence of default costs from a contracting model. Section 4 presents a number of analytical results characterizing optimal default policies. In sections 5 and 6 we then quantitatively evaluate the model predictions by studying optimal default policies in

a setting with business cycle sized shocks and economic disaster shocks, respectively. Section 7 studies the welfare implications of using the default option and section 8 discusses an extension of the model to bonds with longer maturity. A conclusion briefly summarizes. Technical material is contained in a series of appendices.

2 A Small Open Economy Model

Consider a small open economy with shocks to domestic productivity where the government can borrow and invest internationally to insure domestic consumption against fluctuations in domestic income.[4] The economy is populated by a representative consumer with expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

where $c \geq 0$ denotes consumption and $\beta \in (0,1)$ the discount factor. We assume $u(\cdot)$ to be twice continuously differentiable with $u' > 0$ and $u'' < 0$, and that Inada conditions hold. Domestic output is produced by a representative firm using the production function

$$y_t = z_t k_{t-1}^{\alpha} - \bar{c},$$

where y_t denotes output of consumption goods in period t, $\bar{c} \geq 0$ some fixed expenditures, k_{t-1} the capital stock from the previous period, $\alpha \in (0,1)$ the capital share, and $z_t > 0$ an exogenous stochastic productivity shock. Productivity shocks are the only source of randomness in the model and cause domestic income to be risky. Productivity assumes values from some finite set $Z = \left\{ z^1, ..., z^N \right\}$ with $N \in \mathbb{N}$ and the transition probabilities across periods are described by some measure $\pi(z'|z)$ for all $z', z \in Z$. Without loss of generality, we order productivity states such that $z^1 > z^2 > ... > z^N$. The fixed expenditures $\bar{c} \geq 0$ can either be interpreted as an output component that is consumed as a fixed cost in the production process, or - as we prefer - as a fixed subsistence level for consumption expenditures. In the latter case, y_t denotes output in excess of this subsistence level.[5] What is important is that $\bar{c}$ is a domestic output component that cannot be transferred to international lenders to repay outstanding debt. Postulating the existence of such output components appears natural, given that services and other output components are often not tradable. In our quantitative analysis we calibrate $\bar{c} \geq 0$ in a way to obtain reasonably tight international borrowing limits for the domestic economy. All analytical results hold independently of the value chosen for $\bar{c} \geq 0$.

2.1 The Government

The government seeks to maximize the utility of the representative domestic household (1) and is fully committed to its plans. It can insure consumption against

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[4]For the contracting model presented in section 3, it is optimal that the government borrows on behalf of private agents in the international market. Alternatively, one may assume that domestic consumers have no access to the international capital market.

[5]This is consistent with the utility specification in equation (1) if we set $u(c) = -\infty$ for all $c < 0$, i.e. whenever consumption falls short of its subsistence level.

domestic income risk by investing in foreign bonds, i.e. by building up a buffer stock of foreign wealth, and by issuing own bonds, i.e. by borrowing internationally.[6]

Without loss of generality, we consider a setting in which foreign bonds are zero coupon bonds with a maturity of one period.[7] Foreign bonds are assumed to be risk-free and the interest rate r on these bonds satisfies $1 + r = 1/\beta$. We let $F_t \geq 0$ denote the government's holdings of foreign bonds in period t. These bonds mature in period $t + 1$ and repay F_t units of consumption at maturity.

We shall assume that the domestic government has a specific 'technology' available for issuing domestic bonds, i.e., for borrowing internationally. We provide microfoundations for our 'technology' assumptions in section 3 below using an explicit contracting framework.

Specifically, the government can issue non-contingent zero coupon bonds which promise - as part of their legally stated payment obligation - to repay unconditionally one unit of consumption one period after they have been issued. The effects of introducing domestic bonds with longer maturity will be discussed separately in section 8. The government can choose to deviate from the legal payment obligation at maturity, i.e. can choose to default, but such deviations are costly. The 'default costs' take the form of a dead-weight resource cost, capturing the intuitive fact that sizeable *ex post* costs can be associated with a sovereign default event.

Let $D_t \geq 0$ denote the amount of domestic bonds issued by the government in period t. These bonds legally promise to repay D_t units of consumption in period $t + 1$. When issuing these bonds in period t, the government also decides on a default profile $\Delta_t \in [0, 1]^N$, which is a vector determining, for each future productivity state z^n $(n = 1, ..., N)$, what share of the legal payment promise the government will default on:

$$\Delta_t = (\delta_t^1, ..., \delta_t^N).$$

An entry of one indicates a state in which full default occurs, an entry of zero a state with full repayment, and intermediate values capture partial default events. Let $\delta_t(z_{t+1})$ denote the entry in the default profile Δ_t pertaining to productivity state $z_{t+1} \in Z$. Total repayment on domestic bonds maturing in period $t + 1$ is then given by

$$D_t(1 - \delta_t(z_{t+1})) + \lambda D_t \delta_t(z_{t+1}). \tag{2}$$

The first term captures the amount of domestic debt that is repaid to lenders, net of the default share $\delta_t(z_{t+1})$; the second terms captures the default costs accruing to the sovereign borrower, where $\lambda \geq 0$ is a cost parameter. Default costs only emerge if $\delta_t(z_{t+1}) > 0$, occur in the period in which the default takes place, and are assumed to be proportional to the default amount $D_t \delta_t(z_{t+1})$. The analysis would remain

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[6]For the contracting model presented in section 3, it is actually optimal that the government borrows internationally on behalf of private agents. Alternatively, one may assume that private agents do not have access to the international capital markets.

[7]Allowing for a richer maturity structure for foreign bonds makes no difference to the analysis: the small open economy setting implies that foreign interest rates are independent of domestic conditions, such that the government cannot use the maturity structure of foreign bonds to insure against domestic productivity shocks. Foreign bonds thus only serve as a store of value.

unchanged if default costs were instead spread out over time, as long as the present value of the costs still sums to $\lambda D_t \delta_t(z_{t+1})$, when discounting costs at the international interest rate $1 + r$. Proportional default costs are considered mainly for analytical convenience: such a specification allows us to prove concavity of the Ramsey problem later on. While it may be plausible that sovereign default events also give rise to fixed costs that are independent of the default amount, such specifications generate non-convexities in the constraint set of the Ramsey problem, which considerably complicate the optimal policy analysis. Our proportional specification is furthermore similar to the specifications used in Zame (1993) and Dubey, Geanakoplos and Shubik (2005), who previously introduced proportional default costs to study default on private contracts.[8]

In the setting just described, the government can insure domestic consumption against productivity risk either by adjusting its holdings of foreign and domestic bonds, i.e. by adjusting its buffer stock of savings or debt, by choosing appropriate default policies on domestic bonds, or by adjusting domestic investment. The optimal mix between these insurance mechanisms will depend on the level of the default costs λ.

2.2 The Ramsey Problem

To derive the Ramsey problem determining optimal government policies, it turns out to be useful to define the amount of resources available to the domestic government at the beginning of the period, i.e. before issuing new domestic debt, before making investment decisions, and before paying for fixed expenditures, but after (partial) repayment of maturing bonds.[9] We refer to these resources as beginning-of-period wealth and define them as

$$w_t \equiv z_t k_{t-1}^\alpha + F_{t-1} - D_{t-1}(1 - (1 - \lambda)\delta_{t-1}(z_t)). \tag{3}$$

Beginning-of-period wealth is a function of past decisions and of current exogenous shocks only. The government can raise additional resources in period t by issuing new domestic bonds, and use the available funds to invest in foreign riskless bonds, in the domestic capital stock, to finance consumption, and to pay for the fixed expenditures $\bar{c}$. The economy's budget constraint is thus given by

$$w_t + \frac{D_t}{1 + R(z_t, \Delta_t)} = c_t + \bar{c} + k_t + \frac{F_t}{1 + r},$$

where $1/(1 + r)$ and $1/(1 + R(z_t, \Delta_t))$ denote the issue price of the foreign and domestic bond, respectively. The domestic interest rate $R(z_t, \Delta_t)$ depends on the default

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[8]Default costs in our setting represent a resource cost, while the general equilibrium literature with incomplete markets referenced above introduces default cost in the form of a direct utility cost, which enters separably into the borrower's utility function. The resource cost specification is more natural given the microfoundations we provide in section 3, but we conjecture that imposing a direct utility cost would give rise to very similar optimal default implications.

[9]Below we do not distinguish between the government budget and the household budget, instead considering the economy wide resources that are available. This implicitly assumes that the government can costlessly transfer resources between these two budgets, for example, via lump sum taxes.

profile Δ_t chosen by the government and on the current productivity state, as the latter generally affects the likelihood of entering different states tomorrow. Due to the small open economy assumption, the government can take the pricing function $R(\cdot, \cdot)$ as given in its optimization problem. Assuming that international investors are risk-neutral, this pricing function is given by

$$\frac{1}{1 + R(z_t, \Delta_t)} = \frac{1}{1 + r} \sum_{n=1}^{N} (1 - \delta_t(z^n)) \cdot \pi(z^n | z_t), \tag{4}$$

which equates the expected returns on the domestic bond and the foreign bond.

Using the previous notation, the Ramsey problem characterizing optimal government policy is then given by

$$\max_{\left\{F_t \geq 0, D_t \geq 0, \Delta_t \in [0,1]^N, k_t \geq 0, c_t \geq 0\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{5a}$$

$$s.t. :$$

$$c_t = w_t - \overline{c} - k_t + \frac{D_t}{1 + R(z_t, \Delta_t)} - \frac{F_t}{1 + r} \tag{5b}$$

$$w_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \tag{5c}$$

$$w_0, z_0 : given.$$

We impose the natural borrowing limits (5c) on the problem to prevent the possibility of explosive debt dynamics (Ponzi schemes). We allow the natural borrowing limits to be potentially state contingent and assume that the initial condition satisfies $w_0 \geq NBL(z_0)$. The time-zero optimal Ramsey policy involves defaulting on all outstanding debt at time zero, a feature that should be reflected in the initial value for w_0.

While intuitive, the Ramsey problem (5) is characterized by two features that complicate its solution. First, the price of the domestic government bond in the constraint (5b) depends on the chosen default profile, with the result that the constraint fails to be linear in the government's choice variables. It is thus unclear whether problem (5) is concave, which prevents us from working with first order conditions. Second, the presence of the natural borrowing limits (5c) creates problems for numerical solution algorithms. Specifically, imposing sufficiently lax natural borrowing limits, as is usually recommended if one wants to rule out Ponzi schemes only, gives rise to a non-existence problem: sufficiently lax borrowing limits imply that there exist beginning-of-period wealth levels above these limits, for which no policy can insure that the borrowing limits are respected under all contingencies. This non-existence of optimal policies creates problems for numerical solution approaches and thus for a quantitative evaluation of the model. While one could remedy the existence problem by imposing sufficiently tight borrowing limits, this may imply ruling out feasible and potentially optimal policies that would be consistent with non-explosive debt dynamics.

The next sections address both of these issues in turn. We first prove concavity by reformulating problem (5) into a specific variant of a complete markets model, which

can be shown to be concave and equivalent to the original problem. This approach to proving concavity is - to the best of our knowledge - new to the literature and should be useful in a range of other applications involving default decisions. We then proceed by showing how to deal properly with the presence of natural borrowing limits in numerical applications. Again, this approach seems new to the literature and is of interest for a range of other applications. In a final step, we show that the concave and equivalent formulation of the Ramsey problem has a recursive structure, which facilitates numerical solution.

2.2.1 Concavity of the Ramsey Problem

We now define an alternative Ramsey problem with a different asset market structure. As we show, this alternative problem is equivalent to the original problem (5). Since the alternative problem is concave, we can work with first order conditions.

Consider a setting in which the government can trade each period N Arrow securities and a single riskless bond. All assets have a maturity of one period. The vector of Arrow security holdings in period t is denoted by $a_t \in \mathbb{R}^N$ and the n-th Arrow security pays one unit of output in $t+1$ if productivity state z^n materializes. The associated price vector is denoted by $p_t \in \mathbb{R}^N$. Given the risk-neutrality of international lenders, the price of the n-th Arrow security in period t is

$$p_t(z^n) = \frac{1}{1+r} \pi(z^n | z_t).$$ (6)

Let b_t denote the country's holdings of riskless bonds in period t. As before, the interest rate on riskless instruments is $1+r$ and these bonds mature in $t+1$. Beginning-of-period wealth can then be expressed as

$$\widetilde{w}_t \equiv z_t \widetilde{k}_{t-1}^\alpha + b_{t-1} + (1-\lambda) a_{t-1}(z_t),$$ (7)

where $a_{t-1}(z_t)$ denotes the amount of Arrow securities purchased for state z_t, $\widetilde{k}_{t-1}$ capital invested in the previous period, b_{t-1} the bond holdings from the previous period, and $\lambda \geq 0$ the parameter capturing potential default costs in the original problem (5). Note that the Arrow securities in equation (7) pay out only $1-\lambda$ units of consumption to the holder of the asset, but are priced by the issuer in equation (6) as if they would pay one unit of consumption. This wedge will capture the presence of default costs. Since Arrow securities can be used to replicate the payout of the riskless bond, the price system - as perceived by the domestic sovereign - is not arbitrage-free whenever $\lambda > 0$. To have a well-defined problem, we therefore impose the additional constraint $a \geq 0$, while leaving b unconstrained. Intuitively, the restriction $a \geq 0$ insures that the country cannot 'create' additional resources in the form of negative default costs by going short in the Arrow securities.

The Ramsey problem for this alternative asset structure is then given by

$$\max_{\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq 0\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t) \tag{8a}$$

$$s.t.$$

$$\widetilde{c}_t = \widetilde{w}_t - \overline{c} - \widetilde{k}_t - \frac{1}{1+r} b_t - p'_t a_t \tag{8b}$$

$$\widetilde{w}_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z$$

$$\widetilde{w}_0 = w_0, \ z_0 \ given.$$

Problem (8) has the same concave objective function as problem (5) and imposes the same natural borrowing limits. Importantly, however, the constraint (8b) is now linear in the choice variables, as the price vector p_t is independent of domestic policy, so that first order conditions (FOCs) provide necessary and sufficient conditions for optimality.[10] The necessary and sufficient FOCs of problem (8) can be found in Appendix A.1. Appendix A.2 then proves the following equivalence result:

Proposition 1 *A consumption path $\{c_t\}_{t=0}^{\infty}$ is feasible in problem (5) if and only if the consumption path $\{\widetilde{c}_t\}_{t=0}^{\infty}$, with $\widetilde{c}_t = c_t$ for all $t \geq 0$, is feasible in problem (8).*

The proof of proposition 1 shows how the financial market choices $\{b_t, a_t\}$ supporting a consumption allocation in problem (8) can be translated into financial market and default choices $\{F_t, D_t, \Delta_t\}$ supporting the same consumption allocation in the original problem (5), and vice versa. The relationship between these set of choices is given by

$$b_t = F_t - D_t \tag{9}$$

$$a_t = D_t \Delta_t. \tag{10}$$

The riskless bond position b in problem (8) can thus be interpreted as the net foreign asset position in problem (5), while the Arrow security holdings a in problem (8) can be interpreted as the state contingent default decisions on domestic bonds. We will make use of this interpretation in the latter part of the paper, as we solve the simpler problem (8), but interpret the solution in terms of the financial market choices for the original problem (5) with default. Moreover, to support the same consumption allocation in problems (8) and (5) requires identical investment decisions, i.e., $\widetilde{k}_t = k_t$ for all $t \geq 0$, allowing us to use these variables interchangeably.

2.2.2 Dealing with Natural Borrowing Limits

In our quantitative evaluation of the model, we wish to impose borrowing limits that insure the existence of optimal policies, but which are sufficiently lax to not rule out policies that would be consistent with non-explosive debt dynamics. We call such

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[10]This follows from the additional observation that future beginning of period wealth, as defined in equation (7), is a linear function of the financial market choices (a, b) and a convex function of investment k.

borrowing limits the 'marginally binding natural borrowing limits'. We explain below how they can be computed and derive their properties.

Let $NBL(z^n)$ denote the marginally binding natural borrowing limit (NBL) in productivity state z^n, $n = 1, \ldots, N$. It is defined by the following optimization problem[11]

$$NBL(z^n) = \arg\min \widetilde{w}(z^n) \; s.t. \tag{11}$$
$$\widetilde{w}'(z^j) \geq NBL(z^j) \text{ for } j = 1, \ldots, N,$$

where $\widetilde{w}(z^n)$ denotes beginning-of-period wealth in state z^n and $\widetilde{w}'(z^j)$ the beginning-of-period wealth in the next period if next period's productivity is z^j. Marginally binding NBLs can thus be interpreted as a set of state-contingent *minimum* beginning-of-period wealth levels, such that beginning-of-period wealth in all future states remains above these same limits.[12] Problem (11) shows that the marginally binding NBLs are implicitly defined by a fixed point problem.

The fixed point problem (11) is non-trivial because the optimization problem it contains admits a considerable number of corner solutions, due to the presence of linear components in the constraints and objective function, and due to presence of inequality constraints for the choice variables.[13] In numerical solution approaches, it is possible to check all possible corners, each of which gives rise to a set of possible borrowing limits $NBL(z^n)$ ($n = 1, ..., N$) solving the fixed point problem implicitly defined by (11). Although we never encountered such a situation in our numerical applications, it is generally unclear whether there exists a corner solution which provides the *uniformly* lowest borrowing limit across all productivity states z^n ($n = 1, ..., N$). In general, one corner may imply a tighter borrowing limit for one productivity state but a laxer limit for another productivity state than another corner. In such a situation it would be unclear which set of marginally binding NBLs should be imposed. To overcome this potential problem, it is helpful to impose the following mild regularity condition:

Regularity Condition: The productivity process $\pi(\cdot|\cdot)$ is such that lower productivity states are associated with weakly tighter borrowing limits:

$$NBL(z^1) \leq NBL(z^2) \leq ... \leq NBL(z^N) \tag{12}$$

Regularity condition (12) is satisfied, for example, when productivity states are iid or for the polar case where productivity states display sufficiently high persistence.[14] For all of our calibrated productivity processes, we find that the regularity

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[11]A more explicit formulation of the problem is provided in (35) in appendix A.3.

[12]These limits depend only on the current productivity shock because the shock process is Markov and because beginning-of-period wealth is the only other state variable, as will become clear in section 2.2.3.

[13]This can be seen from the more explicit formulation of (11) provided in equation (35) in Appendix A.3.

[14]In the former case, the optimization problems (11) are identical for all states z^n ($n = 1, ...N$),

condition (12) holds.[15] Overall, the regularity condition (12) is of interest because of the following important result:

Proposition 2 *If regularity condition (12) holds, then there generically exists for all model parameterizations a unique solution to the fixed point problem implicitly defined by (11).*

The proof of the proposition can be found in appendix A.3. The proof is constructive, i.e., it also explains how the NBLs can actually be computed. The following proposition then shows that the unique fixed point solution to (11) indeed defines the loosest borrowing limits consistent with non-explosive debt dynamics:

Proposition 3 *Suppose regularity condition (12) holds. Given a productivity state z^n ($n \in \{1, \ldots, N\}$) and a beginning-of-period wealth level $\widetilde{w}$:*

1. *If $\widetilde{w} \geq NBL(z^n)$, then there exists a policy that is consistent with non-explosive debt dynamics along all future contingencies.*

2. *If $\widetilde{w} < NBL(z^n)$ then there exists no policy that does not violate any finite debt limit with positive probability.*

The proof of proposition 3 can be found in Appendix A.4.

2.2.3 Recursive Formulation of the Ramsey Problem

We now show that the Ramsey problem (8) has a recursive structure. This is of interest because it simplifies the representation and characterization of optimal policies. Let $V(\widetilde{w}_t, z_t)$ denote the value function associated with optimal continuation policies when starting with beginning of period wealth $\widetilde{w}_t$ and productivity state z_t. The Ramsey problem (8) then has a recursive representation given by

$$V(\widetilde{w}_t, z_t) = \max_{b_t, a_t \geq 0, \widetilde{k}_t \geq 0} u(\widetilde{w}_t - \bar{c} - \widetilde{k}_t - \frac{1}{1+r}b_t - p_t' a_t) + \beta E_t[V(\widetilde{w}_{t+1}, z_{t+1})]$$
$$s.t. \quad \widetilde{w}_{t+1} = z_{t+1}\widetilde{k}_t^\alpha + b_t + (1-\lambda)a_t(z_{t+1})$$
$$\widetilde{w}_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z,$$

which shows that optimal policies can be expressed as functions of just two state variables $(\widetilde{w}_t, z_t)$.

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such that (12) must hold with strict equality at the fixed point. In the latter case, (12) holds with strict inequality when states are perfectly persistent, due to assumed ordering $z^1 > z^2 > \ldots > z^N$. This continues to be true if the likelihood of transiting into other states is sufficiently small, as buying insurance for such states to satisfy the borrowing limits is then extremely cheap, see equation (6), and will not lead to a reordering of the borrowing limits.

[15]In our numerical applications we also check for possible alternative solutions to (11) which would not satisfy (12).

3   Endogenously Incomplete Government Debt Markets

We now provide explicit microfoundations for the previously made assumptions that the government issues only non-contingent debt and that deviations from the legally stated repayment promise give rise to default costs. We do so by considering a setting where the government can issue arbitrary state contingent debt contracts, but where contracting frictions make it optimal for the government to issue debt with only a non-contingent legal repayment promise. The same frictions also give rise to default costs. The microfoundations we provide below provide a specific example justifying the setup specified in the previous section, but a range of other conceivable microfoundations may exist.

3.1   Explicit and Implicit Contract Components

We consider a setting where a government debt contract consists of two contract components. The first component is the explicit contract, which is written down in the form of a legal text. In its most general form, the legal text consists of a description of the contingencies z^n and of the legal repayment obligations $l^n \geq 0$ associated with each contingency $n \in \{1, ..., N\}$.[16] We normalize the size of the legal contract by assuming $\max_n l^n = 1$. The second component is an implicit contract component. This component is not formalized in explicit terms but is commonly understood by the contracting parties. We capture such implicit contract components by a state contingent 'default profile' $\Delta = (\delta^1, ..., \delta^N) \in [0, 1]^N$, which specifies for each possible contingency the share of the legal payment obligation that is *not* fulfilled by the government.[17] Actual repayment at maturity is then jointly determined by the explicit and implicit contract components and given by

$$l^n(1 - \delta^n)$$

for each contingency $n \in \{1, ..., N\}$. If a contingency arises for which $\delta^n > 0$, the countries pays back less than the legally (or explicitly) specified amount l^n and we shall say that 'the country is in default'. The explicit and implicit contract components are perfectly known to agents.

In the setting just described, a desired state-contingent repayment profile can be implemented by incorporating it either into the explicit legal repayment profile l^n or into the implicit profile δ^n. In the absence of further frictions, these two components would be perfect substitutes and the optimal form of the government debt contract thus indeterminate.

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[16]The fact that $l^n \geq 0$ can be justified by assuming a lack of commitment on the lenders' side. Such a lack of commitment appears reasonable, given the existence of secondary markets on which government debt can be traded.

[17]The fact that $\delta^n \leq 1$ can again be justified by a lack of commitment on the lenders' side, which makes it impossible to write contracts that specify additional transfers to the borrower at maturity. The assumption that $\delta^n \geq 0$ facilitates interpretation in terms of default, but is never binding in our numerical applications.

3.2 Contracting Frictions

We now introduce two simple contracting frictions. First, we assume that explicit legal contracting is costly. This reflects the fact that writing down an explicit legal text describing a financial contract requires the input of lawyers and bankers, thus consuming resources. Second, we assume that implicit contracting, while not creating similar resource costs, gives rise to the risk that the common understanding about the implicit contract component may be lost *after the maturity date* of the contract. This reflects the possibility that agents may have difficulties recalling the implicit contract agreements after a long period of time, especially after the maturity date of the contract. This differs for the explicit contract, where agents can always go back and read about their contract obligations, such that common understanding is insured independently of time. The fact that the common understanding about the implicit contract components may disappear is perfectly and rationally anticipated by all agents.

We now describe these two frictions in greater detail. We normalize the costs of writing a non-contingent legal contract ($l^n = 1$ for $n = 1, ..., N$) to zero and assume that incorporating a contingency gives rise to a proportional legal fee $\lambda \geq 0$ that is charged against the value of the contingent agreement. This is in line with the casual empirical observation that lawyers typically charge fees that are proportional to the value of the agreements they formulate. In particular, legally incorporating a payment $l^n \leq 1$ for some contingency z^n in the explicit contract, involves the costs

$$\lambda \left(1 - l^n\right)$$

per contract issued, where $1 - l^n$ denotes the value of the deviation from the baseline payment of 1.

While incorporating a state contingency in the repayment structure via the implicit contract component $\delta^n > 0$ does not give rise to costs, it exposes the government to the risk that the common understanding about a default event may be lost after the maturity date of the contract. This is relevant for the borrower because in the absence of a recallable implicit contract component, courts base their decisions on a comparison of the explicit contract obligation with the actual actions (payments) that occurred. Default events that are followed by a lack of common understanding about the implicit contract thus provide strong incentives for lenders to sue the government for fulfilment of the explicit contract, i.e. to sue the government for repayment of the legally stated amount.[18] Anticipating such behavior, the government will engage - at the time the default occurs - in a negotiation process with the lender, with the objective to reach an *explicit* legal settlement that protects it from being sued in the future.

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[18] As documented in Panizza, Sturzenegger and Zettelmeyer (2009), legal changes in a range of countries in the late 1970s and early 1980s eliminated the legal principle of 'sovereign immunity' when it came to sovereign borrowing. Specifically, in the U.S. and the U.K. private parties can sue foreign governments in courts if the complaint relates to a commercial activity, amongst which courts regularly count the issuance of sovereign bonds. We implicitly assume that lenders cannot commit to not sue the government. Again, this appears plausible, given that secondary markets allow initial buyers of government debt to sell the debt instruments to other agents.

The settlement agreement transforms the previously only implicit contract component into an explicit one, by stating that the debt contract is regarded as fulfilled, even if the actual payment amount fell short of the amount specified in the legal text of the contract. The threat of going to court to obtain such an explicit settlement via a court ruling in the period where the default happens and where a common understanding about the implicit components still exists, will induce the lender to agree to such an agreement.[19] Since we assume explicit legal contracting to be costly, the settlement agreement following a default event gives rise to the legal costs (or default costs)

$$\lambda l^n \delta^n$$

per contract, where $l^n \delta^n$ denotes the value of the settlement agreement, i.e. the defaulted amount on each contract. For simplicity, we assume here that the same proportional fee λ is charged in the ex-post settlement stage as applies when writing an explicit contingent contract *ex ante*. We discuss below the case where the ex-post settlement costs are higher. While the legal fees associated with writing a legal contract are assumed to be borne by the government, we allow for the possibility that the settlement fees are shared between the lender and the borrower, with the lender paying $\lambda^l \geq 0$, the borrower paying $\lambda^b \geq 0$, and $\lambda^l + \lambda^b = \lambda$.

3.3 Optimal Government Debt Contract

Consider a government that wishes to implement a contingent payment $p(z) \leq 1$ for some contingency $z \in Z$. Specifying the contingency as part of the legal contract involves the contract writing costs

$$\lambda \left(1 - p(z) \right)$$

per contract and no *ex post* settlement costs in case the default contingency arises in the future. Alternatively, not specifying the contingent payment as part of the legal contract gives rise to expected default costs of[20]

$$\Pr(z|z_0)\lambda \left(1 - p(z) \right),$$

where z_0 is the contingency prevailing at the time when the contract is issued. Since $\Pr(z|z_0) \leq 1$ and since default costs are borne at a later stage, i.e. when the contract matures, the government will always strictly prefer to issue a non-contingent explicit contract and to shift contingencies into the implicit contract profile. This continues to be true even in the more general case where the *ex post* settlement costs are much higher than the cost associated with incorporating the contingency *ex ante* into

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[19] The fact that - due to the large number of actors involved - the implicit contract component of government debt can be verified in court makes government debt contracts special. Implicit components of private contracts, for example, are often private information available to the contracting parties only, and thus cannot be verified in court, not even over the lifetime of the contract. The optimal form of private contracts, therefore, generally differs from the optimal form of government debt contracts.

[20] The expected settlement cost for the lender enters the borrower's optimization reasoning because the borrower has to compensate the lender *ex ante* for the expected costs borne by the lender.

the legal contract, provided the probability $\Pr(z|z_0)$ of reaching the default event is sufficiently small.

Summing up, it is optimal for the government to issue debt that is non-contingent in explicit legal terms. At the same time, the government has the option to deviate from the legally specified payment amount, but such actions give rise to proportional default costs λ. We thus provided microfoundations for the assumptions entertained in the previous section.

3.4 Government versus Private Debt

The contracting framework can also be used to justify why the government optimally borrows on behalf of private agents in the international market. This is highly relevant because it implies that private and sovereign default cannot be used interchangeably to insure domestic consumption. The optimality of sovereign borrowing arises because of a fundamental difference between sovereign and private debt contracts: the implicit contract components of private contracts are private information to the contracting parties, and thus cannot be verified in court, not even over the lifetime of the contract. This is different for a sovereign debt contract that is widely shared among many individuals. Achieving state contingency in private contracts thus has to rely on explicit contracting, which creates costs that dominate those incurred in sovereign borrowing, as the non-contingency of sovereign bonds is optimal precisely to avoid the costs of explicit legal contracting.

4 Optimal Sovereign Default: Analytic Results

This section presents a number of analytic results characterizing the optimal default policies that solve the Ramsey problem (8). We first consider - for benchmark purposes - a setting without default costs ($\lambda = 0$). As we show, the full repayment assumption is then suboptimal under commitment and sovereign default is optimal for virtually all productivity realizations. This holds true independently of the country's net foreign asset position. We then show that for 'prohibitive' default cost levels with $\lambda \geq 1$, default is never optimal. Finally, we present analytic results for the most relevant case with intermediate levels of default costs ($0 < \lambda < 1$). Ramsey optimal default decisions then depend on the country's wealth level and on the productivity realization. Interestingly, there exists a discontinuity in the optimal default policies as one moves from $\lambda = 0$ to $\lambda > 0$.

4.1 Zero Default Costs

In the absence of default costs ($\lambda = 0$) the original Ramsey problem (5) reduces to a generalized version of the problem analyzed in section II in Grossman and Van Huyck (1988).[21] The proposition below shows that - as in Grossman and Van Huyck - full consumption smoothing and frequent default are optimal. The proof of the proposition can be found in Appendix A.5.

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[21]Grossman and Van Huyck consider an endowment economy with iid income risk, which is a special case of our setting with production and potentially serially correlated productivity shocks.

Proposition 4 *For $\lambda = 0$ the solution to the Ramsey problem (8) involves constant consumption equal to*

$$\widetilde{c} = (1 - \beta)(\Pi(z_0) + \widetilde{w}_0) \tag{13}$$

where $\Pi(\cdot)$ denotes the maximized expected discounted profits from production, defined as

$$\Pi(z_t) \equiv E_t \left[\sum_{j=0}^{\infty} \beta^j \left(-k^*(z_{t+j}) + \beta z_{t+j+1} \left(k^*(z_{t+j}) \right)^{\alpha} - \overline{c} \right) \right]$$

with

$$k^*(z_t) = (\alpha \beta E(z_{t+1}|z_t))^{\frac{1}{1-\alpha}} \tag{14}$$

denoting the optimal investment policy. For any period t, the optimal default level satisfies

$$a_{t-1}(z_t) \propto - \left(\Pi(z_t) + z_t \left(k^*(z_{t-1}) \right)^{\alpha} \right) \tag{15}$$

Let $N_t \subset \{1, ..., N\}$ denote the number of productivity states in t that can be reached from z_{t-1} in $t-1$, according to the transition matrix $\Pi(\cdot|\cdot)$. Since $a_{t-1} \geq 0$, it follows from equation (15) that the optimal commitment policy generically involves default for at least $N_t - 1$ productivity realizations in t.[22] Default thereby insures the domestic economy against two sources of risk: first, it insures against a low realization of current output due to a low value of current productivity, as captured by the term $z_t \left(k^*(z_{t-1}) \right)^{\alpha}$ in equation (15), a risk that is present in similar form in the endowment setting of Grossman and Van Huyck (1988); second, it additionally insures the domestic economy against (adverse) news regarding the expected profitability of future investments, as captured by the term $\Pi(z_t)$. As a result of this policy the 'net worth' of the economy, defined as the sum of expected future profits $\Pi(z_t)$ and accumulated net wealth $\widetilde{w}_t$, remains constant over time and equal to its initial value $\Pi(z_0) + \widetilde{w}_0$.[23] In the absence of default costs, risk sharing thus fully and exclusively occurs via optimal sovereign default, with net worth remaining constant over time, and domestic investment being at it expected profit-maximizing level (14).

To interpret the optimal default patterns implied by proposition 4, suppose that expected future profits $\Pi(z_t)$ are weakly increasing with current productivity z_t. This is the case whenever z_t is a sufficiently persistent process, but also if z_t is iid so that expected future profits are independent of current productivity. Equation (15) then implies that optimal default levels are inversely related to the current level of productivity, i.e. the amount of non-repaid claims strictly increases with the distance of current productivity from its maximal level. This pattern is optimal independently of the wealth level of the economy, i.e. is optimal even if the economy has a positive net foreign asset position. With a positive net foreign asset position, the sovereign optimally issues domestic bonds and invests the proceeds into foreign bonds, so as to

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[22]Default is not required for states z_t achieving the maximal value for $\Pi(z_t) + z_t(k^*(z_{t-1}))^{\alpha}$ across all $z_t \in Z$. For such states, default can be set equal to zero, with default levels being strictly positive for all other states. This, however, is not the only possible default pattern implementing full consumption stabilization: it would also be possible to choose strictly positive default levels for the states z_t achieving the maximal value for $\Pi(z_t) + z_t(k^*(z_{t-1}))^{\alpha}$.

[23]This follows from the proof of proposition 4 in appendix A.5.

be able to default on the domestic bonds following adverse shocks, see equations (9) and (10).

4.2 Prohibitive Default Costs

We now consider the polar case with prohibitive default cost levels $\lambda \geq 1$. Default events then induce deadweight resource costs that (weakly) exceed the amount of resources that the borrower does not repay to lenders. Net of default costs, the sovereign thus cannot gain resources by defaulting. For the equivalent Ramsey problem (8) this implies that the payout from Arrow securities is weakly negative, while the price of Arrow securities for states that can be reached with positive probability is strictly positive, see equation (6). This leads to the following obvious result:[24]

Lemma 1 *For $\lambda \geq 1$ it is optimal to choose $a_t = 0$ for all t.*

For $\lambda \geq 1$ it is thus optimal never to use default to insure domestic consumption. Instead insurance occurs via the accumulation and decumulation of non-contingent and non-defaultable bonds and potentially via adjustments of the investment margin.[25] An interesting trade-off between default and the adjustment of non-defaultable bond positions thus emerges for a plausible range of intermediate cost specifications with $0 < \lambda < 1$, as investigated in the next section.

4.3 Intermediate Default Cost Levels

Deriving an analytic solution for the Ramsey policy problem (8) for arbitrary intermediate default cost levels $0 < \lambda < 1$ is generally difficult, as the Ramsey problem is a non-linear dynamic stochastic optimization problem with an endogenous state variable and a number of occasionally binding inequality constraints. Nevertheless, it is feasible to derive analytic solutions to the Ramsey problem when beginning-of-period wealth is either very low or very high. More precisely, we consider below the case where beginning-of-period wealth level is at its lower bound, i.e. at the marginally binding natural borrowing limit defined in section 2.2.2. In a second step, we derive an approximation to the Ramsey policy that applies when beginning-of-period wealth is sufficiently high. Our results show that the country's wealth position is an important determinant of optimal default policies, in contrast to what is suggested by the limiting case with zero or prohibitive default costs.

4.3.1 Initial Wealth at Its Lower Limit

Consider first a situation where beginning-of-period wealth is at its lower bound ($\widetilde{w}_t = NBL(z_t)$). We can then define a critical productivity state index n_t^*:

$$n_t^* = \arg \max_{n \in [1,...,N]} n \tag{16}$$

$$s.t. \sum_{i=n}^{N} \pi(z^i | z_t) \geq 1 - \lambda$$

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[24]The Arrow security choices for future states that are reached with zero probability do not affect welfare, which allows us to set them also equal to zero.

[25]The latter is discussed in detail in the next section.

The critical index n_t^* is defined as the highest productivity index, such that reaching states z^n tomorrow with $n \geq n_t^*$ still has a likelihood larger than $1 - \lambda$. It turns out that the index n_t^* divides states for which default is Ramsey optimal tomorrow from states for which full repayment is optimal. The critical index also affects the optimal investment level and the optimal amount of bond holdings. The following proposition characterizes the optimal policy in period t as a function of the critical index n_t^* :[26]

Proposition 5 *Suppose regularity condition (12) holds and $\widetilde{w}_t = NBL(z_t)$. Then the optimal policy solving the Ramsey policy problem (8) in period t is given by*

$$\widetilde{k}_t = \left(\alpha \beta \left(\frac{\sum_{n=1}^{n_t^*} \pi(z^n | z_t) - \lambda}{1 - \lambda} z^{n_t^*} + \frac{\sum_{n=n_t^*+1}^{N} \pi(z^n | z_t) z^n}{1 - \lambda} \right) \right)^{\frac{1}{1-\alpha}} \tag{17}$$

$$b_t = NBL(z^{n_t^*}) - z^{n_t^*} \widetilde{k}_t^\alpha \tag{18}$$

$$\widetilde{c}_t = 0 \tag{19}$$

$$a_t(z^n) = 0 \text{ for } n \leq n_t^* \tag{20}$$

$$a_t(z^n) = \frac{NBL(z^n) - z^n \widetilde{k}_t^\alpha - b_t}{(1 - \lambda)} > 0 \text{ for } n > n_t^* \tag{21}$$

The proof of the proposition is given in Appendix A.6.[27] Equations (20) and (21) jointly show that default is suboptimal for all sufficiently good productivity states z^n with $n \leq n_t^*$. For $n > n_t^*$ strictly positive default is optimal and the default amount is strictly increasing in n, i.e., there is more default the lower is realized productivity. Consistent with earlier results, it will never be optimal to default as $\lambda \to 1$, as then $n_t^* \to N$. Conversely, it is optimal to default in all but one states as $\lambda \to 0$, as then $n_t^* \to 1$. Moreover, it follows from problem (16) that an increase in default costs (weakly) reduces the set of states for which default is optimal.

Proposition 5 also shows that consumption is at its lower bound once wealth is at the marginally binding borrowing limit.[28] Consumption will stay at its lower bound in the next period if a sufficiently bad productivity state is reached, i.e. a state z^n with index $n \geq n_t^*$. This is so because the Arrow security and bond purchases insure that tomorrow's beginning-of-period wealth levels are exactly at their state-contingent marginally binding natural borrowing limit for all productivity states z^n with $n \geq n_t^*$, such that proposition 5 applies again in the next period. Yet, if a productivity state z^n with $n < n^*$ is reached, then beginning-of-period wealth will strictly exceed the natural borrowing limit and consumption will move back to strictly positive values. Obviously, this can happen only if λ is sufficiently large, as otherwise $n_t^* = 1$.

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[26] For λ sufficiently close to 1 or $\lambda > 1$ we have $n_t^* = N$, such that all expressions in proposition 5 remain well-defined.

[27] Note that the proposition determines optimal policy in period t, when beginning-of-period wealth is at its lower bound, but does not determine the optimal policy for other periods. This is possible, even if the underlying optimization problem is a dynamic infinite horizon problem, because the choice set at the marginally binding natural borrowing limit reduces to a singleton.

[28] Total consumption can still be positive, as we allow for a positive subsistence level of consumption expenditures $\bar{c} \geq 0$.

Equation (17) shows that the presence of default costs also distorts the optimal investment decision downward relative to the expected profit-maximizing investment level $k^*(z_t)$ defined in (14). This downward distortion is (ceteris paribus) increasing in n_t^* and in the default costs λ. Since n_t^* is itself an increasing function of λ, the total effect of increased default cost is to decrease investment relative to the efficient level. Intuitively, lower investment reduces the output risk generated by productivity shocks, as these affect the existing capital stock multiplicatively, and this helps to satisfy the marginally binding natural borrowing limits.

4.3.2 Large Wealth Levels

This section derives analytical expressions that approximate the Ramsey optimal policies for sufficiently large beginning-of-period wealth levels. The following position summarizes the main result:

Proposition 6 *Suppose* $\lambda > 0$ *and consumption preferences satisfy* $u''(c) \to 0$ *as* $c \to \infty$. *Consider a time horizon* $T < \infty$ *and for* $j = 0, ..., T$ *the policies*

$$
\begin{aligned}
\widetilde{c}_{t+j} &= (1 - \beta)(\Pi(z_{t+j}) + \widetilde{w}_{t+j}) \\
\widetilde{k}_{t+j} &= k^*(z_{t+j}) \\
b_{t+j} &= (1 + r)\left(\widetilde{w}_{t+j} - k^*(z_{t+j}) - \widetilde{c}_{t+j} - \overline{c}\right) \\
a_{t+j}(z^n) &= 0 \text{ for all } n = 1, ..., N
\end{aligned}
$$

where $\Pi(z_{t+j})$ *and* $k^*(z_{t+j})$ *are as defined in proposition 4. For any* $\epsilon > 0$ *we can find a wealth level* $\bar{w} < \infty$ *so that for all finite initial wealth levels* $\widetilde{w}_t \geq \bar{w}$, *the Euler equation errors* e_{t+j} *implied by the policies above satisfy* $e_{t+j} < \epsilon$ *for all periods* $j = 0, ..., T - 1$.

The proof of the proposition is contained in Appendix A.7. The fact that the Euler equation error vanishes for sufficiently large initial wealth levels $\widetilde{w}_t$ implies that the policies stated in the proposition approximate the truly optimal policies increasingly well; see Santos (2000), for example. For a sufficiently large wealth level, domestic investment thus remains undistorted, i.e. maximizes expected discounted profits. In addition, it is optimal always to fully repay debt. This is in stark contrast to the case with $\lambda = 0$ where frequent default is optimal, independently of the wealth level, see proposition 4. There thus exists a discontinuity of optimal default policies at $\lambda = 0$. The discontinuity implies that for sufficiently high wealth levels, the presence of even tiny default costs implies a complete shift from frequent default to no default, such that risk sharing occurs optimally only via self-insurance, i.e. via the accumulation and decumulation of international wealth. This result is true as long as u'' decreases towards zero as consumption increases without bond. This is the case for commonly used preference specifications, e.g. preferences with constant relative risk aversion or constant absolute risk aversion. Intuitively, vanishing curvature causes the output risk of given size implied by optimal investment levels to have only negligible influence on consumption utility, whenever consumption (and thus wealth) are sufficiently high. Since default costs are strictly positive, it then becomes suboptimal to use default to insure against these output fluctuations.

5 Quantitative Exploration of Optimal Default Policy

This section investigates whether sovereign default can be optimal in a realistically calibrated version of the model with business cycle sized shocks to domestic productivity. The theoretical results established in the previous section show that sovereign default is suboptimal for sufficiently high wealth levels, but likely to be optimal for low wealth levels. This section analyzes optimal default policies for intermediate wealth levels and determines how high the economy's net foreign asset position plausibly has to be, so as to cause sovereign default to be suboptimal. To answer this quantitative question, we solve for the optimal default policies numerically.[29]

The next section presents the model calibration, including our estimation of default costs. The resulting optimal default policies are discussed subsequently.

5.1 Model Calibration

We interpret a model period as one year. For the productivity process we use a standard parameterization from the business cycles literature and set the annual persistence of technology to $(0.9)^4$ and the annual standard deviation for the innovation to 1%.[30] Using Tauchen's (1986) procedure to discretize into a process with two states, one obtains a high productivity state $z^h = 1.0133$, a low productivity state $z^l = 0.9868$ and a transition matrix

$$\pi\left(\cdot|\cdot\right) = \begin{pmatrix} 0.8077 & 0.1923 \\ 0.1923 & 0.8077 \end{pmatrix}. \tag{22}$$

The capital share parameter in the production function is set to $\alpha = 0.34$ and the annual discount factor to $\beta = 0.97$. The latter implies an annual real interest rate of approximately 3% for risk-free debt instruments. We choose consumption preferences with constant relative risk aversion

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

and a moderate degree of risk aversion by setting $\sigma = 2$. The preference specification satisfies the assumption in proposition 6. We calibrate the fixed expenditures $\bar{c}$ that cannot be transferred to foreign lenders such that, if the government is forced to repay debt in all contingencies, the marginally binding NBLs imply that the net foreign asset position of the country cannot fall below -100% of average GDP in any productivity state.[31] We thereby seek to capture the fact that industrialized countries, which

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[29]Determining the Ramsey optimal policies requires solving a non-linear stochastic dynamic optimization problem involving occasionally binding inequality constraints. Appendix A.10 explains how this has been achieved.

[30]The quantitative results reported below are not very sensitive to the precise numbers used. A corresponding calibration at a quarterly frequency is employed in Adam (2011).

[31]Average GDP is defined as the average output level associated with efficient investment, i.e., when $k_t = k^*(z_t)$ each period, as defined in proposition 4. We thereby average over the ergodic distribution of the z process. For our parameterization, this yields an average output level of 0.5647 before fixed expenditures. Furthermore, at the marginally binding NBLs, government decisions are exlusivley determined by the desire to prevent debt from exploding, such that the marginally binding NBL can indeed be used to calibrate the model. The resulting level of fixed expenditures is given by $\bar{c} = 0.3540$.

default only rarely, virtually never have a net foreign asset position below -100% of GDP, see figure 10 in Lane and Milesi-Ferretti (2007).[32] It thus appears plausible to consider a calibration implying that countries cannot sustain higher external debt levels without running the risk of a sovereign default.

It only remains to determine the default cost parameter λ. While default (or contracting) costs are notoriously difficult to estimate, it is possible to exploit restrictions from our structural model to obtain an estimated lower bound for the costs of default. The idea underlying our estimation approach is to exploit the fact that default costs that accrue to the lender (but not those borne by the borrower) can be estimated from financial market prices and information on default events. This is feasible because the borrower has to compensate the lender *ex ante* for the default costs arising on the lender's side, such that lenders' default costs are reflected in financial market prices and can thus be backed out from these.

To exploit this idea, we consider a slightly more general setting than that considered in the Ramsey problem (5), namely one where the lender also bears default costs ($\lambda^l > 0$). Total default costs are then given by $\lambda = \lambda^l + \lambda^b$ with λ^b denoting the borrower's default cost. The structure of the Ramsey problem (5) then remains unchanged, except for the bond pricing equation (4), which has to be adapted so as to reflect the presence of the lenders' default costs λ^l:[33,34]

$$\frac{1}{1 + R(z_t, \Delta)} = \frac{1}{1+r} \sum_{n=1}^{N} \left(1 - (1 + \lambda^l)\delta^n\right) \cdot \pi(z^n | z_t) \qquad (23)$$

Appendix A.9 shows how the previous equation can be combined with data on *ex post* returns from Klingen, Weder, and Zettelmeyer (2004), who consider 21 countries over the period 1970-2000, and data on default events for the corresponding set of countries and years from Cruces and Trebesch (2011), to obtain an estimate of the lender's default cost.[35] This yields

$$\lambda^l = 6.1\%$$

and suggests that lenders suffer a loss of about 6% of the default amount in a sovereign default event. Note that this loss is in addition to the direct losses that result from incomplete repayment by the sovereign debtor: for every dollar that is not repaid to the lender, the lender suffers an additional loss of 6 cents.

The total costs of default include the costs accruing to the lender and to the borrower. Therefore, we consider in our quantitative analysis default cost levels λ

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[32]Three out of the five industrialized countries approaching this boundary in 2004 later faced fiscal solvency problems (Greece, Portugal and Iceland).

[33]In the definition of the beginning of period wealth level (3), λ has to replaced by λ^b, where the latter captures the borrower's proportional default costs.

[34]Appendix A.8 shows that if a consumption allocation is feasible in a setting where default costs are borne exclusively by the borrower, then it is also feasible if some or all of these costs are borne by the lender instead, as long as the total amount of default costs λ remains unchanged.

[35]Cruces and Trebesch kindly provided us with required information for the set of countries and years considered in Klingen, Weder, and Zettelmeyer (2004).
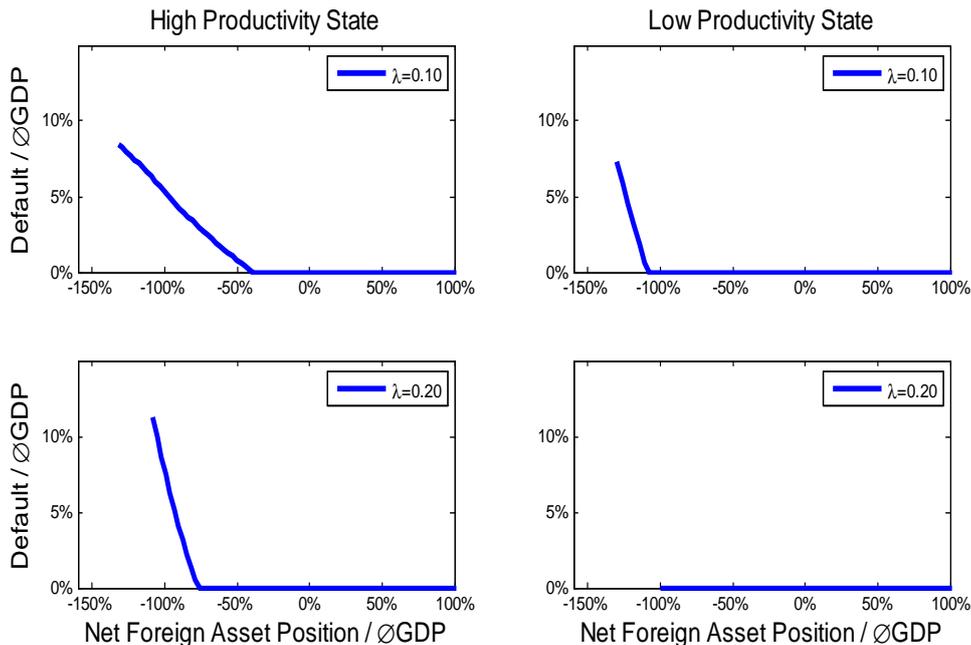
Figure 1: Optimal Default Policies (top row: λ=10%, bottom row: λ=20%)

that exceed the estimated value of λ^l. Specifically, we shall consider default cost levels of 10% and 20%, respectively. A value of $\lambda = 10\%$ implies that about 60% of the overall default costs are borne by the lender. A setting with $\lambda = 20\%$ is less conservative and implies that slightly more than two-thirds of the total default costs accrue on the borrower's side.

5.2 Optimal Default with Business Cycle Shocks to Productivity

This section determines the optimal default policies for our calibrated model from the previous section. The top and bottom rows of figure 1 depict the optimal default policies for $\lambda = 10\%$ and $\lambda = 20\%$, respectively. Graphs on the left show the optimal default policy when current productivity is high (z^h), while graphs on the right depict the policy if current productivity is low (z^l). Each graph of the figure reports the optimal default amount in the next period if productivity in that period is low (z^l). Default is optimally zero, whenever the high productivity state materializes tomorrow. To facilitate interpretation, the optimal default policies are depicted as a function of the net foreign asset position of the country, which is shown on the x-axis.[36] Moreover, the default amounts (y-axis) and the net foreign asset positions (x-axis) are both normalized by average GDP. A value of -100% on the x-axis, for example, corresponds to a situation where the government has issued repayment claims

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[36]The net foreign asset position b_t, defined in section 2.2.1, is a strictly increasing function of the state variable $\widetilde{w}_t$, allowing us to substitute $\widetilde{w}_t$ by b_t in the graphs when depicting optimal policies.

such that its net foreign asset position equals -100% of average GDP. Since the net foreign asset position and beginning-of-period wealth commove positively in the optimal solution, the lowest value of the net foreign asset position for which policies are depicted corresponds to the point where the country's beginning-of-period wealth level has reached its marginally binding borrowing limit.

Figure 1 shows that default is the more desirable, the lower is the country's net foreign asset (or wealth) position and the lower are the default costs. Overall, however, default is suboptimal over a wide range of net foreign asset positions. For $\lambda = 20\%$, for example, default is always suboptimal tomorrow, whenever today's productivity is low (z^l). If today's productivity is high, then default is optimal tomorrow only if the net foreign asset position is close to its lowest level, i.e. if wealth is close to its marginally binding borrowing limit.

For $\lambda = 0$ the optimal default policies are flat and strictly positively valued functions (this follows from proposition 4, which shows that default policies do not depend on wealth); figure 1 thus shows that even moderate levels of default costs cause default to be suboptimal over a large range of net foreign asset positions. The assumption of full repayment, standardly entertained in the Ramsey literature with incomplete markets, appears to provide - at first glance - a reasonable approximation to the fully optimal Ramsey policy.

Yet, as we show next, optimal default decisions have the potential to significantly increase welfare relative to an economy where default is ruled out by assumption, especially if the country's international wealth position is low. To illustrate this fact, let c_t^1 denote the optimal state contingent consumption path in an economy where default is ruled out by assumption, and c_t^2 the corresponding consumption path with (costly) optimal default decisions. We then compute the welfare equivalent consumption increase ω, which causes the path c_t^1 to be welfare equivalent to path c_t^2 over the first 500 years[37], which is given by

$$E_0 \left[\sum_{t=0}^{500} \beta^t \frac{((c_t^1 + \omega(c_t^1 + \bar{c}))^{1-\gamma}}{1-\gamma} \right] = E_0 \left[\sum_{t=0}^{500} \beta^t \frac{(c_t^2)^{1-\gamma}}{1-\gamma} \right], \tag{24}$$

where the expectations are evaluated by averaging over 10,000 sample paths. Note that the consumption increase ω is measured against total domestic expenditures, which also includes the fixed expenditures $\bar{c}$. This leads to a potential understatement of the consumption equivalent welfare gain, but appears sensible given that the variable consumption component c_t^1 approaches zero as the beginning of period wealth level approaches the marginally binding natural borrowing limit.

Since default is never optimal for high wealth levels, the achievable welfare gain depends on the initial net foreign asset position, with lower wealth positions giving rise to higher welfare gains. Similarly, lower default costs also give rise to higher welfare gains. An upper bound for the welfare gains can thus be computed by considering a low value for the plausible range of default costs, say $\lambda = 10\%$, and a low value for the initial wealth position. The lowest sustainable wealth position, when default

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[37]We choose a finite horizon to evaluate the welfare implications because the net foreign asset position is not necessarily stationary under the optimal policy.
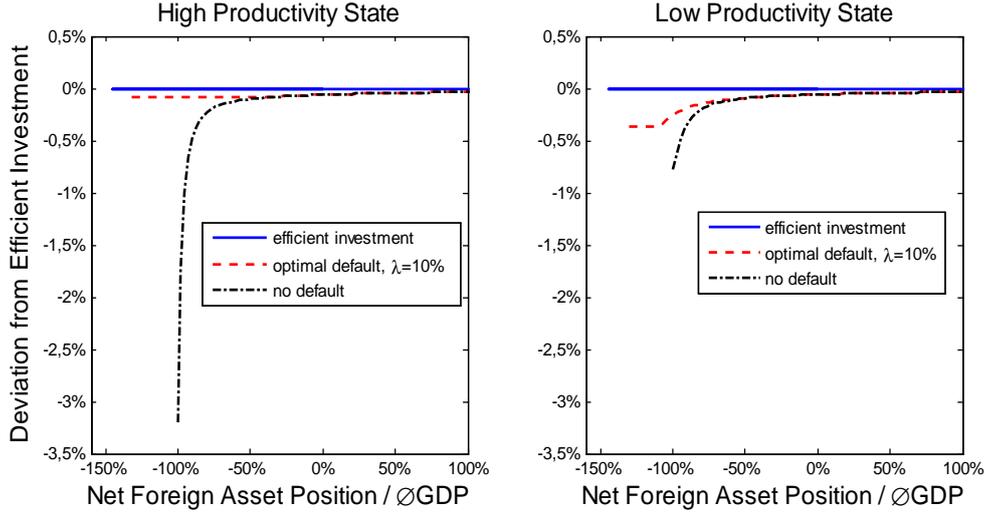
Figure 2: Investment Distortions With and Without Default

is ruled out, is the one for which the net foreign asset position reaches -100% of GDP. At this point, the welfare gains are sizeable and amount to $\omega = 1.9\%$ of total consumption expenditures each period when $\lambda = 10\%$. For a somewhat higher initial wealth position, for example, the one implying a net foreign asset position of -80% of GDP, the welfare gains largely disappear: we then have $\omega = 0.01\%$.

For three reasons, welfare increases strongly at the marginally binding borrowing limit of the no-default economy: First, allowing for default relaxes the borrowing limits, as we show in the next paragraph, thereby allowing a better smoothing of consumption. Second, default provides - via the conditional resource transfers - additional insurance for consumption. Third, the possibility of default increases the efficiency of investment. Figure 2 illustrates this last point by comparing optimal investment levels for the case with default (assuming $\lambda = 10\%$) and without default. Investment is depicted in deviation from the efficient investment level[38] and as a function of the country's net foreign asset position. The figure shows that the investment margin in the no-default economy is strongly distorted as the net foreign asset position approaches -100% of GDP.[39]

Figure 2 reveals that the investment distortions are much smaller if default is chosen optimally. This occurs because the ability to default following low productivity realizations relaxes the marginally binding natural borrowing limits, as is illustrated in figure 3. The figure depicts - again for both productivity states - how the net foreign asset position at the marginally binding borrowing constraint depends on the level of default costs (depicted on the x-axis). For $\lambda = 100\%$, which is a setting where

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[38]The efficient investment level is the expected profit-maximizing level defined in equation (14).

[39]Indeed, as shown in proposition 5, at the borrowing limit, investment in the no-default case is optimally given by the one that would be efficient if the lowest possible productivity state is expected to materialize for sure tomorrow. Given the persistence of the high and low productivity states, this explains why the investment distortion is larger in the high productivity state.
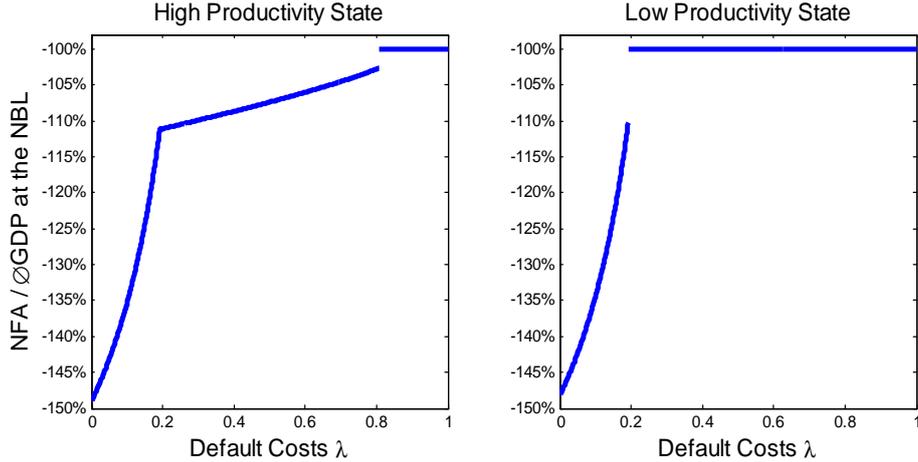
Figure 3: The Effect of Default Costs on the Sustainable NFA Positions

default is suboptimal (or simply ruled out), the maximally sustainable net foreign asset position equals -100%, as implied by our calibration. For more reasonable default cost levels, say in the range of 10% to 20%, the possibility to default strongly relaxes the sustainable net foreign asset position. This relaxation allows the economy to carry more output risk and thus to sustain higher investment levels.[40]

6 Optimal Default with Economic Disasters

The previous section showed that, for plausible default cost levels, it is suboptimal to default on government debt in a setting with business cycle shocks, provided the country is not too close to its borrowing limit. This section quantitatively evaluates to what extent this conclusion continues to be true in a setting with much larger economic shocks. Consideration of larger shocks is motivated by the observation that countries occasionally experience very large negative shocks, as argued by Rietz (1988) and Barro (2006), and that such shocks tend to be associated with a government default in the data.[41] To capture the possibility of large shocks, we augment the model by including disaster-like shocks to aggregate productivity and then explore the quantitative implications of disaster risk for optimal sovereign default decisions.

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[40]The two discontinuities in the net foreign asset positions present in figure 3 relate to two default cost thresholds, above which default in the low state tomorrow becomes suboptimal. Specifically, the first discontinuity arises at $\lambda = 19.23\%$, which equals one minus the transition probability from z^l to z^l. Above this cost level, it becomes suboptimal to default in the next period if current productivity is low. The second discontinuity arises at $\lambda = 80.77\%$, which equals one minus the probability of transiting from z^h into z^l. Above this cost level, it also becomes suboptimal to default if the current state is high. For our calibrated model, the no-default assumption is thus fully optimal whenever $\lambda \geq 80.77\%$. More generally, the no-default assumption is consistent with optimality whenever default costs exceed a level of one minus the probability of reaching an undesirable state. In settings where the probability of reaching an undesirable state is very low, default costs thus need to be close to the prohibitive level of 100% to make full repayment Ramsey optimal in all states.

[41]Barro (2006) and Gourio (2010) also consider sovereign default in disaster states. Due to the different focus of their analysis, default probabilities and default rates are exogenous in their setting.

6.1 Calibrating Disasters

To capture economic disasters we introduce two disaster-sized productivity levels to our aggregate productivity process. We add two disaster states rather than a single state to capture the idea that the size of economic disasters is uncertain *ex ante*. The inclusion of two disaster states also allows us to calibrate the disaster shocks in a way that they match both the mean and the variance of GDP disaster events.

Using a sample of 157 GDP disasters, Barro and Jin (2011) report a mean reduction in GDP of 20.4% and a standard deviation of 12.64%. Assuming that it is equally likely to enter both disaster states from the 'normal' business cycle states (z^h, z^l), the productivity levels $z^d = 0.9224$ and $z^{dd} = 0.6696$ are obtained for a medium-sized disaster and for a severe disaster, respectively. Our vector of possible productivity realizations thus takes the form $Z = \left\{ z^h, z^l, z^d, z^{dd} \right\}$, where the parameterization of the business cycle states $\left(z^h, z^l \right)$ is the same as in section 5.1.

The state transition matrix for the shock process is described by the matrix

$$
\pi = \begin{pmatrix}
0.7770 & 0.1850 & 0.019 & 0.019 \\
0.1850 & 0.7770 & 0.019 & 0.019 \\
0.1429 & 0.1429 & 0.3571 & 0.3571 \\
0.1429 & 0.1429 & 0.3571 & 0.3571
\end{pmatrix},
$$

where the transition probability from the business cycle states into the disaster states is chosen to match the unconditional disaster probability of 3.8% per year, as reported in Barro and Jin (2011). We thereby assume that both disaster states are reached with equal likelihood. The persistence of the disaster states is set to match the average duration of GDP disasters, which equals 3.5 years. For simplicity, we assume that conditional on staying in a disaster, the medium and severe disaster states are equally likely to materialize. Finally, the transition probabilities of the business cycle states (22) are rescaled to reflect the transition probability into a disaster state.

We also recalibrate the subsistence level for consumption $\bar{c}$. As in section 5, we choose $\bar{c}$ such that, in an economy where bonds must always be repaid, the economy can sustain a maximum net foreign debt position of 100% of average GDP in the business cycle states (z^h, z^l).[42]

6.2 Optimal Default with Disasters: Quantitative Analysis

Figure 4 depicts the optimal default policies for the economy with disaster shocks when $\lambda = 10\%$. Each panel in the figure corresponds to a different productivity state today and reports the optimal amount of default for tomorrow's states z^l, z^d and z^{dd}, respectively. As before, default is never optimal if the highest productivity state z^h realizes in the next period. The default amounts (on the y-axis) are depicted as a function of the country's net foreign asset position (on the x-axis of each graph), with both variables being normalized by average GDP.

Figure 4 reveals that, in the presence of large shocks, the introduction of optimal repayment decisions tremendously relaxes the marginally binding borrowing limits,

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[42]This yields an adjusted value of $\bar{c} = 0.1899$. Choosing tighter limits does not affect the shape of the optimal default policies but only shifts the policies depicted in figure 4 'to the right'.
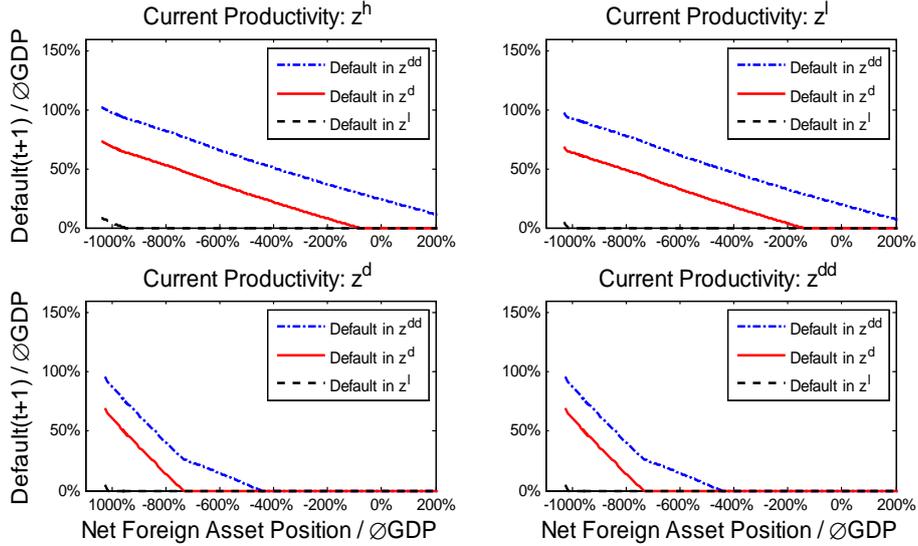
Figure 4: Optimal Default Policies with Disaster States ($\lambda = 10\%$)

much more than occurs in a setting with small shocks only. For the considered calibration, the sustainable net foreign asset position at the borrowing limit drops from -100% of GDP for the case where repayment must always occur to a level of about -1,000% of GDP. Figure 4 also shows that it is virtually never optimal to default in the low business cycle state (z^l), unless the net foreign asset position is very close to its maximally sustainable level, similar to section 5 where we considered business cycle shocks only. Furthermore, for a wide range of values for the net foreign asset position, it is optimal to default if the economy makes a transition from a business cycle state to a disaster state; see the top panels in the figure. Default is optimal for a transition to the severe disaster state (z^{dd}), even when the country's net foreign asset position is positive before the disaster.[43] Default is thus optimal even if the country has sufficient resources (holdings of foreign debt) to repay its own debt. Nevertheless, the optimal default amount increases as the country's net foreign asset position worsens.

Once the economy is in a disaster state, a further default in the event that the economy remains in the disaster state is optimal only if the net foreign asset position is very low, see the bottom panels of figure 4. Since the likelihood of staying in a disaster state is quite high, choosing not to repay if the disaster persists would have large effects on interest rates *ex ante*. As a result, serial default in case of a persistent disaster is optimal only if the net foreign asset position is sufficiently negative.

At the end of 2011 the net international investment position of the United States stood at -26.7% of GDP.[44] Assuming that the U.S. was in the low business cycle

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[43]With a positive net foreign asset position, default can occur because the country chooses to issue sufficient amounts of own debt, which are invested in the foreign bond, with the sole purpose of being able to default on domestic debt if a transition to a severe disaster occurs.

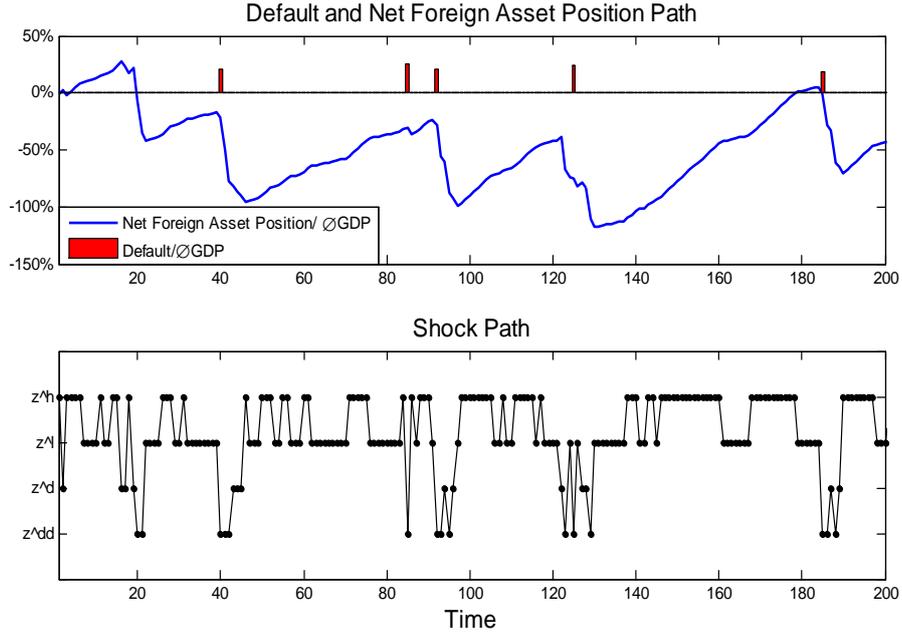[44]See data provided by the Bureau of Economic Analysis.

Figure 5: Net Foreign Asset Dynamics under Optimal Default Policy ($\lambda = 10\%$)

state (z^l) and that the net international investment position can be approximated by the model-implied net foreign asset position, figure 4 suggests that the U.S. should default on its outstanding debt to the order of 21% of GDP, if it enters a severe economic disaster (z^{dd}), which is a situation where its GDP level falls approximately 30% below trend. In dollar terms such a default would amount to 3.16 trn. At the same time, figure 4 suggests that a U.S. default is suboptimal in the case of a medium sized output disaster (z^d), which implies an approximate shortfall of GDP of 8% below trend. Default for the medium sized disaster state becomes optimal only for considerably more negative net foreign asset positions, namely below -83% of GDP in the high business cycle (z^h) and below -147% of GDP in low state (z^l).

The overall shape of the optimal default policies is fairly robust to assuming different values for the default costs λ. Larger default costs, for example, shift the default policies 'towards the left', i.e. default occurs only for more negative net foreign asset positions. Larger default costs also tighten the maximally sustainable net foreign asset positions, thereby reducing the range of net foreign asset positions over which optimal policies exist.

The top panel of figure 5 reports a typical sample path for the net foreign asset position and the amount of default induced by the optimal default policies shown in figure 4. The panel on the bottom of figure 5 depicts the productivity shock process. The economy starts at a zero net foreign asset position and each model period corresponds to one year. The figure shows that it is optimal to improve the net foreign asset position when the economy is in the business cycle states, with faster improvements being optimal in the high business cycle state; see, for example, the
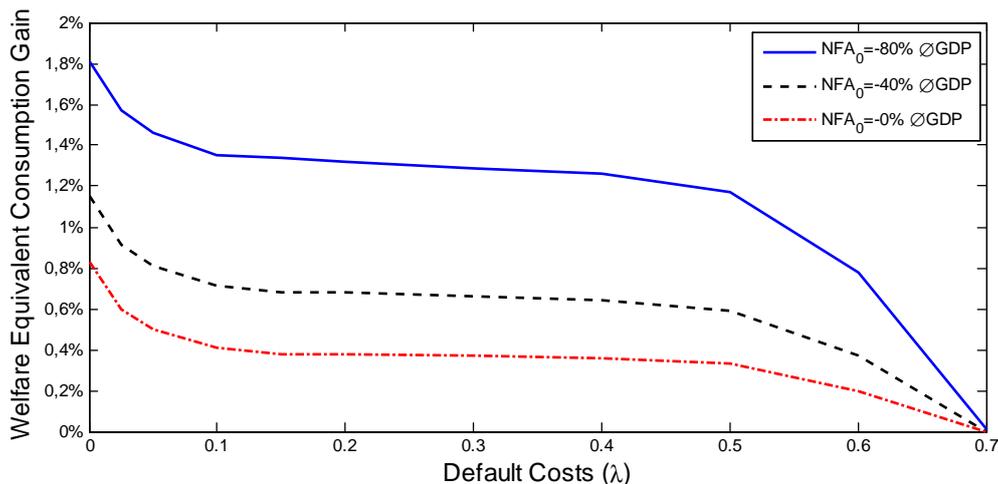
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Figure 6: Welfare Gains from Using the Default Option

50-year period without economic disasters starting in year 130. Over this period the net foreign asset position improves from a value slightly below -100% of GDP to small positive values.

Figure 5 also shows that a transition to the severe disaster state usually leads to default, provided the transition does not occur via the medium disaster state first (as is the case for year 20). Also, following a disaster, the net foreign asset position deteriorates, whenever the disaster persists for more than one period (see for example year 40), otherwise the net foreign asset position is largely unaffected or even improves slightly (see year 86). This shows that, for the considered level of default cost, default provides only partial insurance against disaster risk. At the same time, default can be optimal even if the net foreign asset position is positive, see period 186. As a result of the optimal default policies, the net foreign asset dynamics are typically characterized by rapid deteriorations during persistent disaster periods and gradual improvements during normal times, with the latter accelerating during high business cycles states.

7 Welfare Gains from Optimal Default

This section documents that optimal default gives rise to sizeable welfare gains when the economy is subject to potential output disasters, as considered in section 6. Specifically, we determine the welfare equivalent consumption gains relative to a situation where repayment of debt is simply assumed always to occur, as defined in equation (24). As we show, the welfare gains are surprisingly robust to the assumed level of default costs, as long as these do not reach prohibitive levels.

Figure 6 shows how the consumption equivalent welfare gains depend on the default costs (shown on the x-axis). The figure depicts these gains for three different initial wealth positions at time zero, which imply initial net foreign asset positions of 0%, -40% and -80% of GDP, respectively. Depending on the initial wealth position, the welfare gains amount - for a broad range of default costs - to 0.5-1.5%
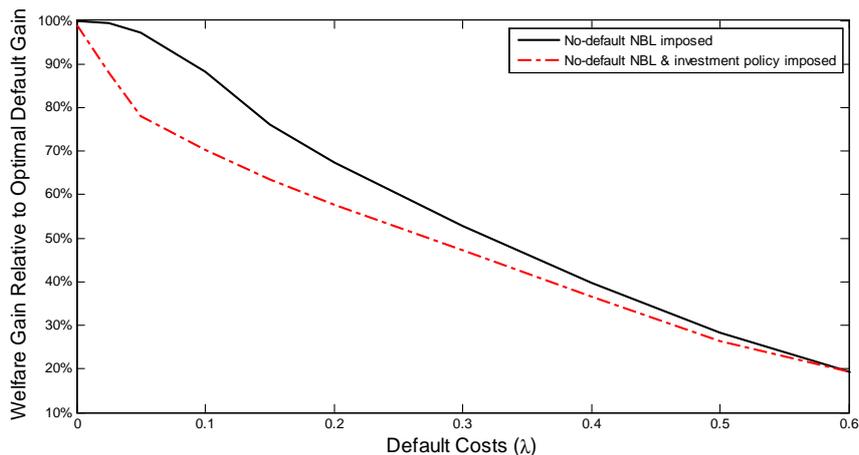
Figure 7: Decomposition of the Welfare Gains

of consumption each period, with the welfare gains increasing further at low default cost levels. Overall, the welfare gains are surprisingly robust over a wide range of default costs, instead proving more sensitive to the assumed initial net foreign asset position. For default costs $\lambda \geq 0.5$ the welfare gains from default decrease steeply and for $\lambda \geq 0.7$ welfare gains largely disappear. For such default cost levels it becomes suboptimal to insure against a future disaster state when the economy is already in a disaster, independently of the country's net foreign asset position. As a result, the natural borrowing limits tighten significantly in the disaster states[45] and the required amount of insurance in the business cycle state (z^l, z^h) increases strongly as the net foreign asset position deteriorates. While some insurance via default is still optimal for such high cost levels, it does not give rise to sizeable welfare gains as default is very costly.

Figure 7 decomposes the welfare gains shown in figure 6 into three sources.[46] In a first step, we compute optimal default policies but impose the borrowing limits associated with the no-default economy. The solid line in figure 7 shows that for very low levels of default costs, we still capture almost all welfare gains, while for higher default costs, the relaxation of the marginally binding borrowing limits accounts for the majority of the welfare gains. Intuitively, for sufficiently low default costs the country's wealth position is rather stable, as default insures against adverse shocks, which means that it is unlikely that the borrowing limit of the no-default economy is ever reached. This is different for higher default cost levels, where the country's wealth position fluctuates considerably; see, for example, figure 5. In a second step, we impose the borrowing limits from the no-default economy and the investment policy that is optimal in the no-default regime. The achievable welfare gains are depicted by the dashed line in figure 7. The difference between the solid and dashed lines captures the share of welfare gains associated with improved investment policies

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[45]They reach the levels applying in the economy with non-defaultable bonds.

[46]We assume an initial net foreign asset position of -80% as implied by the upper line in figure 6.

in the default economy. For plausible levels of the default costs, say, in the range of 10-20%, improved investment policies account for about 10-20% of the total welfare gains, but are relatively small otherwise. Finally, the dashed line in figure 7 indicates that insurance via default accounts for the majority of the welfare gains for low levels of default costs, but that this share decreases as default costs rise. Intuitively, this happens because default becomes less frequent when default costs rise, with the relaxation of the borrowing limits then accounting for the majority of the welfare gains.

8 Long Maturities and Optimal Bond Repurchase Programs

This section briefly discusses the effects of also considering bonds with longer maturities.

Introducing longer maturities of risk-free foreign debt has no implications for the prior analysis, as such bonds cannot be used in better ways to insure against domestic output risk. Like short-dated foreign bonds, they merely provide a store of value. Issuing long-dated domestic bonds, however, can offer an advantage over issuing just short bonds, whenever the market value of long bonds reacts to domestic conditions in a way that allows the government to insure more efficiently against domestic shocks than a default on short-dated bonds. It would be desirable, for example, that the market value of outstanding long bonds decreases following a disaster shock, so that the government realizes a capital gain that lowers the overall debt burden. Such capital gains, however, do not materialize if repayment is assumed to occur in all future states, e.g., as in Angeletos (2002). The depreciation of the debt's market value can thus only be induced via the anticipation of future default on long-dated bonds.

Issuing long bonds will offer an advantage against the outright default on maturing short bonds, whenever the costs associated with repurchasing outstanding long bonds at the devaluated market price are lower than the costs induced by a default event on maturing short bonds. If both costs are identical, i.e. if the repurchase in a situation where default is anticipated in the future is associated with the same costs as a default on maturing bonds, then there will be no additional value associated with issuing long bonds. The same holds true if the costs of a repurchase are higher than those of an outright default. When these costs are lower, however, there then exists an advantage from issuing long-dated bonds. Consider, for example, the situation where a repurchase of long bonds does not give rise to any costs. The government will then find it optimal to fully insure domestic consumption, i.e., achieve the first best allocation, independently of the costs associated with an outright default on maturing bonds. The optimal bond issuance strategy will then have the feature that the government issues each period long bonds that (partially) default at maturity, depending on the productivity realization in some earlier period. The default at maturity needs to be calibrated such that the interim capital gains which realize fully insure domestic consumption against domestic productivity shocks, i.e. satisfy the proportionality restriction (15). The government can then repurchase the existing stock of long bonds, avoid default costs entirely, and issue new long bonds with a new

implicit contingent repayment profile, so as to insure against future shocks. In this way outright default on maturing bonds never occurs.

9   Conclusions and Outlook

In a setting with endogenously incomplete government bond markets, debt default is part of Ramsey optimal policy. Allowing the planner to choose whether or not to repay maturing debt relaxes the country's borrowing limits, thereby increasing international risk sharing and the efficiency of domestic investment. This leads to significant welfare gains, even when default costs are sizeable.

Interestingly, the normative benchmark derived in the present analysis also has the potential to offer a plausible positive description of the actually observed debt and default patterns. The present setting with commitment has, for example, no difficulties in rationalizing the existing sovereign debt levels or in generating default patterns where default occurs following negative output realizations. The present approach may thus have an advantage relative to limited commitment approaches, which often face difficulties in explaining the existing sovereign debt levels[47], and struggle with the fact that default incentives are strongest following a positive innovation to domestic income.[48] Exploring the positive relevance of the presented normative theory of optimal default thus remains an interesting avenue for future research. This is especially true because the present analysis shows that Ramsey optimality is entirely consistent with a country choosing not to repay debt in a situation where it has sufficient resources to do so.

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[47]See, for example, the borrowing constraints in tables 2 and 3 reported in the quantitative analysis of Arellano and Heathcote (2010).

[48]The threat of a market exclusion is then least deterring, which prompts Arellano (2008), for example, to introduce state dependent default costs, which tilt the default profile towards low output states.

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A   Appendix

A.1   First Order Equilibrium Conditions

This appendix derives the necessary and sufficient first order conditions for problem (8). Using equation (7) to express beginning-of-period wealth, the problem is given by

$$\max_{\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq 0\}} E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t)$$

$$s.t. :$$

$$\widetilde{c}_t = z_t \widetilde{k}_{t-1}^\alpha + b_{t-1} + (1-\lambda)a_{t-1}(z_t) - \overline{c} - \widetilde{k}_t - \frac{1}{1+r}b_t - p_t' a_t$$

$$z_{t+1}\widetilde{k}_t^\alpha + b_t + (1-\lambda)a_t(z_{t+1}) \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z$$

$$\widetilde{w}_0 = w_0, \ z_0 \ given,$$

We now formulate the Lagrangian Λ, letting η_t denote the multiplier on the budget constraint in period t, $\nu_t(z^n)$ the multiplier for the short-selling constraint on the Arrow security that pays off in state z^n in $t+1$, and $\omega_t \in R^N$ the vector of multipliers associated with the natural borrowing limits for each possible realization of productivity in $t+1$, where $w_t(z^n)$ denotes the entry of the vector pertaining to productivity state z^n:

$$\Lambda = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t) \right.$$

$$+ \beta^t \eta_t \left(-\widetilde{c}_t + z_t \widetilde{k}_{t-1}^\alpha + b_{t-1} + (1-\lambda)a_{t-1}(z_t) - \overline{c} - \widetilde{k}_t - \frac{1}{1+r}b_t - p_t' a_t \right)$$

$$+ \beta^t \sum_{i=1}^{N} \nu_t(z^n) a_t(z^n)$$

$$\left. + \beta^t \sum_{i=1}^{N} \omega_t(z^n) \left(z^n \widetilde{k}_t^\alpha + b_t + (1-\lambda)a_t(z^n) - NBL(z^n) \right) \right]$$

We drop the inequality constraints for $\widetilde{k}_t$ and $\widetilde{c}_t$, as the Inada conditions guarantee an interior solution for these variables whenever $\widetilde{w}_0 > NBL(z^0)$.[49] Differentiating the

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[49] This assumes that the fixed expenditures $\overline{c}$ are truly fixed and accrue independently of the investment level, i.e. even if $k_t = 0$.

Lagrangian with respect to the choice variables, one obtains

$$\widetilde{c}_t : \quad u'(\widetilde{c}_t) - \eta_t = 0$$

$$b_t : \quad -\eta_t \frac{1}{1+r} + \beta E_t \eta_{t+1} + \sum_{n=1}^{N} \omega_t(z^n) = 0$$

$$a_t(z^n) : \quad -\eta_t p_t(z^n) + \beta \pi(z^n|z^t)\eta_{t+1}(z^n)(1-\lambda)$$
$$+\nu_t(z^n) + \omega_t(z^n)(1-\lambda) = 0 \quad \forall n \in N$$

$$\widetilde{k}_t : \quad -\eta_t + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t \eta_{t+1} z_{t+1} + \alpha \widetilde{k}_t^{\alpha-1} \sum_{i=1}^{N} \omega_t(z^n) z^n = 0$$

Using the FOC for consumption to replace η_t in the last three FOCs, delivers three Euler equations:

$$Bond: -u'(\widetilde{c}_t)\frac{1}{1+r} + \beta E_t u'(\widetilde{c}_{t+1}) + \sum_{n=1}^{N} \omega_t(z^n) = 0 \tag{26a}$$

$$Arrow: -u'(\widetilde{c}_t)p_t(z^n) + \beta \pi(z^n|z_t)u'(\widetilde{c}_{t+1}(z^n))(1-\lambda)$$
$$+\nu_t(z^n) + \omega_t(z^n)(1-\lambda) = 0 \quad \forall n \in N \tag{26b}$$

$$Capital: -u'(\widetilde{c}_t) + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t u'(\widetilde{c}_{t+1})z_{t+1} + \alpha \widetilde{k}_t^{\alpha-1} \sum_{i=1}^{N} \omega_t(z^n)z^n = 0 \tag{26c}$$

In addition, we have the following complementarity conditions $\forall n \in N$:

$$0 \leq a_t(z^n), \ \nu_t(z^n) \geq 0, \ \text{one holds strictly} \tag{26d}$$

$$0 \leq z^n \widetilde{k}_t^\alpha + b_t + (1-\lambda)a_t(z^n) - NBL(z^n), \ \omega_{t+1}(z^n)) \geq 0, \ \text{one holds strictly} \tag{26e}$$

Combined with the budget constraint, the Euler equations and the complementarity conditions constitute the necessary and sufficient optimality conditions for problem (8).

A.2 Proof of Proposition 1

Consider some state-contingent beginning-of-period wealth profile w_t arising from some combination of bond holdings, default decisions and capital investment $(F_{t-1}, D_{t-1}, \Delta_{t-1}, k_{t-1})$ in problem (5). We show below that one can generate the same state contingent beginning-of-period wealth profile $\widetilde{w}_t = w_t$ in problem (8) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing an appropriate investment profile (a_{t-1}, b_{t-1}). Moreover, the same amount of funds are required to purchase (a_{t-1}, b_{t-1}) in $t-1$ than to purchase (F_{t-1}, D_{t-1}) when the default profile is Δ_{t-1}. With the costs of financial investments generating a particular future payout profile being the same in both problems, identical physical investments, and identical beginning of period wealth profiles, it then follows from constraints (5b) and (8b) that a consumption path which is feasible in (5) is also feasible in (8).

To simplify notation we establish the previous claim for the case with $N = 2$ productivity states only. The extension to more states is straightforward. Consider

the following state-contingent initial wealth profile

$$\begin{pmatrix} w_t(z^1) \\ w_t(z^2) \end{pmatrix} = \begin{pmatrix} z^1 k_{t-1}^\alpha + F_{t-1} - D_{t-1}(1 - (1-\lambda)\delta_{t-1}(z^1)) \\ z^2 k_{t-1}^\alpha + F_{t-1} - D_{t-1}(1 - (1-\lambda)\delta_{t-1}(z^2)) \end{pmatrix}.$$

As is easily verified, this beginning-of-period wealth profile in problem (8) can be replicated by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing the portfolio

$$b_{t-1} = F_{t-1} - D_{t-1}, \tag{27}$$

$$a_{t-1} = \begin{pmatrix} D_{t-1}\delta_{t-1}(z^1) \\ D_{t-1}\delta_{t-1}(z^2) \end{pmatrix} \tag{28}$$

The funds f_{t-1} required to purchase and issue (F_{t-1}, D_{t-1}) under the default profile $\Delta_{t-1} = (\delta_{t-1}(z^1), \delta_{t-1}(z^2))$ are given by

$$f_{t-1} = \frac{1}{1+r} F_{t-1} - \frac{1}{1 + R(z_{t-1}, \Delta_{t-1})} D_{t-1}$$

where the interest rate satisfies

$$\frac{1}{1 + R(z_{t-1}, \Delta_{t-1})} = \frac{1}{1+r}\left((1 - \delta_{t-1}(z^1))\pi(z^1|z_{t-1}) + (1 - \delta_{t-1}(z^2))\pi(z^2|z_{t-1}) \right).$$

The funds $\widetilde{f}_{t-1}$ required to purchase (b_{t-1}, a_{t-1}) are given by

$$\widetilde{f}_{t-1} = \frac{1}{1+r}(F_{t-1} - D_{t-1}) + \frac{1}{1+r}\left(\delta_{t-1}(z^1)\pi(z^1|z_{t-1}) + \delta_{t-1}(z^2)\pi(z^2|z_{t-1}) \right)G_{t-1}^S,$$

where we used the price of the Arrow security in (6). As can be easily seen $\widetilde{f}_{t-1} = f_{t-1}$, as claimed. This completes the proof that a consumption path which is feasible in (5) is also feasible in (8). Since $a \geq 0$, the reverse is also true, because equations (27) and (28) can then be solved for values $(D_{t-1}, F_{t-1}, \Delta_{t-1})$ satisfying $D_{t-1} \geq 0$, $F_{t-1} \geq 0$, and $\Delta_{t-1} \in [0,1]^N$, so that it is possible in problem (5) to obtain a portfolio with the same contingent payouts.[50] Again, this portfolio has the same costs and thus admits the same consumption choices. This completes the equivalence proof.

A.3 Proof of Proposition 2

Let us define a critical future productivity state n_t^*

$$n_t^* = \arg \max_{n \in [1,\dots,N]} n \tag{29}$$

$$s.t. \sum_{i=n}^{N} \pi(z^i|z_t) \geq 1 - \lambda$$

---

[50]Note that for $a < 0$, no such choices would exist, which shows that $a \geq 0$ is required to obtain equivalence.

and choices

$$\widetilde{k}_t = \left(\alpha\beta \left(\frac{\sum_{n=1}^{n_t^*} \pi(z^n|z_t) - \lambda}{1-\lambda} z^{n*} + \frac{\sum_{n=n_t^*+1}^{N} \pi(z^n|z_t)z^n}{1-\lambda} \right) \right)^{\frac{1}{1-\alpha}} \tag{30}$$

$$b_t = NBL(z^{n_t^*}) - z^{n_t^*}\widetilde{k}_t^{\alpha} \tag{31}$$

$$c_t = 0 \tag{32}$$

$$a_t(z_t^n) = 0 \text{ for } n \leq n_t^* \tag{33}$$

$$a_t(z^n) = \frac{NBL(z^n) - z^n\widetilde{k}_t^{\alpha} - b_t}{(1-\lambda)} > 0 \text{ for } n > n_t^* \tag{34}$$

We now show that these choices satisfy the necessary and sufficient first order conditions of problem (11). Problem (11) can more explicitly be written as

$$\min_{a_t, b_t, \widetilde{k}_t, \widetilde{c}_t} \bar{c} + \widetilde{c}_t + \widetilde{k}_t + \frac{1}{1+r} b_t + \sum_{i=1}^{N} a_t(z^n) p_t(z^n) \tag{35}$$

$$s.t.$$

$$z^n\widetilde{k}_t^{\alpha} + b_t + (1-\lambda)a_t(z^n) - NBL(z^n) \geq 0 \text{ for } n = 1, ..., N$$

$$a_t(z^n) \geq 0 \text{ for } n = 1, ..., N$$

$$\widetilde{c}_t \geq 0$$

Obviously, it is optimal to choose $\widetilde{c}_t = 0$, so that we can eliminate this variable and the constraint $\widetilde{c}_t \geq 0$ from the problem. Letting $\omega_t(z^n)$ denote the Lagrange multipliers for the first set of constraints in (35) and $\nu_t(z^n)$ the multipliers for the second set of constraints, the first order necessary conditions are given by

$$\widetilde{k}_t : 1 + \alpha\widetilde{k}_t^{\alpha-1} \sum_{n=1}^{N} \omega_t(z^n)z^n = 0 \tag{36}$$

$$b_t : \frac{1}{1+r} + \sum_{n=1}^{N} \omega_t(z^n) = 0 \tag{37}$$

$$a_t(z^n) : p_t(z^n) + \omega_t(z^n)(1-\lambda) + \nu_t(z^n) = 0 \tag{38}$$

We also have the constraints

$$z^n\widetilde{k}_t^{\alpha} + b_t + (1-\lambda)a_t(z^n) - NBL(z^n) \geq 0, \omega_t(z^n) \leq 0, \text{ one holding strictly} \tag{39}$$

$$a_t(z^n) \geq 0, \nu_t(z^n) \leq 0, \text{ one holding strictly} \tag{40}$$

Conditions (38) and (40) can equivalently be summarized as

$$p_t(z^n) + \omega_t(z^n)(1-\lambda) \geq 0, a_t(z^n) \geq 0, \text{ one holding strictly} \tag{41}$$

so that the first order conditions are given by (36),(37), (39) and (41). Since the objective is linear and the constraint set convex, the first order conditions are necessary

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and sufficient. We now show that the postulated solution satisfies these first order conditions.

Since $a(z^n) > 0$ for $n > n_t^*$ for the conjectured solution, condition (41) implies

$$
\begin{aligned}
\omega_t(z^n) &= -\frac{p_t(z^n)}{1-\lambda} \\
&= -\frac{1}{1+r}\frac{\pi(z^n|z_t)}{1-\lambda} < 0 \quad \text{for all } n > n_t^*
\end{aligned}
$$

We now conjecture (and verify later) that

$$
\omega_t(z^n) = 0 \quad \text{for all } n < n_t^* \tag{42}
$$

$$
\omega_t(z^{n_t^*}) = -\frac{1}{1+r} - \sum_{n=n^*+1}^{N} \omega_t(z^n) \tag{43}
$$

For the previous conjecture (37) holds by construction. Also, the second inequality of (39) holds for all $n \in \{1, ..., N\}$ because we have

$$
\sum_{n=n_t^*+1}^{N} \omega_t(z^n) = \frac{1}{1+r}\frac{\sum_{n=n_t^*+1}^{N}\pi(z^n|z_t)}{1-\lambda} < \frac{1}{1+r}
$$

from the definition of n^*, so that $\omega_t(z^{n_t^*}) < 0$. Equations (33) and (34) then imply that (41) hold. Furthermore, (30) implies that (36) holds. It thus only remains to show that the first inequality for (39) also holds. For $n \geq n_t^*$ this follows from (34). For $n < n_t^*$ this is also true because $-NBL(z^n)$ is increasing as n falls due to (12), output $z^n \tilde{k}_t^\alpha$ is equally increasing as n falls due to the assumed ordering of the productivity levels, and the first inequality of (39) holds with equality for $n = n_t^*$ due to (31). As a result, the first inequality in (39) holds strictly for $n < n_t^*$, justifying our conjecture (42). This proves that all first order conditions hold for the conjectured solution (30)-(34).

Since the solution (30)-(34) is linear in the NBLs showing up in the constraints of (11), the minimized objective is also a linear function of these NBLs. The fixed point problem defined by (11) is thus characterized by a system of equations that is linear, which generically admits a unique solution.

A.4   Proof of Proposition 3

We start by proving point 2 in proposition 3. Suppose that in some period t and for some productivity state z^n ($n \in \{1, ..., N\}$), beginning-of-period wealth falls short of the limits implied by the marginally binding NBL, i.e.

$$
\widetilde{w}_t(z^n) = NBL(z^n) - \varepsilon, \tag{44}
$$

for some $\varepsilon > 0$. We then prove below that for at least one contingency z^j in $t+1$, which can be reached from z^n in t with positive probability, it must hold that

$$
\widetilde{w}_{t+1}(z^j) \leq NBL(z^j) - \varepsilon(1+r), \tag{45}
$$

such that along this contingency the distance to the marginally binding NBL is increasing at the rate $1 + r > 1$ per period. Since the same reasoning also applies for future periods, and since the marginally binding NBLs assume finite values, this implies the existence of a path of productivity realizations along which wealth far in the future becomes unboundedly negative, such that any finite borrowing limit will be violated with positive probability.

It remains to prove that if (44) holds in period t and contingency z^n, this implies that (45) holds for some contingency z^j in $t + 1$ ($j \in \{1, ..., N\}$) and that z^j can be reached from z^n with positive probability. Suppose for contradiction that

$$\widetilde{w}_{t+1}(z^h) > NBL(z^h) - \varepsilon(1 + r) \tag{46}$$

for all $h \in \{1, ..., N\}$ that can be reached from z^n, i.e. for which $\pi(z^h|z^n) > 0$. The cost-minimizing way to satisfy the constraints (46) for all h, when replacing the strict inequality by a weak one, is to choose the solution (30), (32)- (34) and $b_t = NBL(z^{n_t^*}) - z^{n_t^*} \widetilde{k}_t^\alpha - \varepsilon$.[51] Achieving this requires $NBL(z^n) - \varepsilon$ units of funds in t, which in turn implies that satisfying constraints (46) with strict inequality requires strictly more funds than are available in t, whenever the state can be reached with positive probability. As a result, (45) must hold for at least one j that can be reached from z^n with positive probability. This concludes the proof of point 2 in the proposition.

The proof of point 1 is relatively straightforward. If $\widetilde{w}_t(z^n) \geq NBL(z^n)$, then it is feasible to choose the policies (30), (31), (33), (34) and a non-negative consumption level in period t. This will give rise to beginning-of-period wealth levels in the future that are equal to or above the marginally binding NBLs, and this policy can be repeated in all future periods. Since the marginally binding NBLs assume finite values, this proves the existence of a policy with non-explosive debt dynamics.

A.5   Proof of Proposition 4

We first show that the proposed consumption solution (13) and investment policy (14) satisfy the budget constraint, that the inequality constraints $a \geq 0$ are not binding, and that the NBLs are not binding either. Thereafter, we show that the remaining first order conditions of problem (8), as derived in appendix A.1, hold.

We start by showing that the portfolio implementing (13) in period $t = 1$ is consistent with the flow budget constraint and $a \geq 0$. The result for subsequent periods follows by induction. In period $t = 1$ with productivity state z^n, beginning-of-period wealth under the investment policy (14) is given by

$$\widetilde{w}_1^n \equiv z^n \left(k^*(z_0) \right)^\alpha + b_0 + a_0(z^n) \tag{47}$$

To insure that consumption can stay constant from $t = 1$ onwards we again need

$$\widetilde{c} = (1 - \beta)(\Pi(z^n) + \widetilde{w}_1^n) \tag{48}$$

---

[51]To verify this just solve the minimization problem (35) where the constraints on future wealth are replaced by (46) with the strict inequality repaced by a weak one.

for all possible productivity realizations $n = 1, ..N$. This provides N conditions that can be used to determine the $N + 1$ variables b_0 and $a_0(z^n)$ for $n = 1, ..., N$. We also have the condition $a_0(z^n) \geq 0$ for all n and by choosing $\min_n a_0(z^n) = 0$, we obtain one more condition that makes it possible to pin down a unique portfolio (b_0, a_0). Note that the inequality constraints on a do not bind for the portfolio choice, as we have one degree of freedom, implying that the multipliers $v_1(z^n)$ in Appendix A.1 are zero for all $n = 1, ..., N$. It remains to be shown that the portfolio achieving (48) is feasible given the initial wealth $\widetilde{w}_0$. Using (47) to substitute $\widetilde{w}_1^n$ in equation (48) we obtain

$$\widetilde{c} = (1 - \beta)(\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n)) \qquad \forall n = 1, ...N.$$

Combining with (13) we obtain

$$\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n) = \Pi(z_0) + \widetilde{w}_0$$

Multiplying the previous equation with $\pi(z^n|z_0)$ and summing over all n one obtains

$$E_0\left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + \sum_{n=1}^{N} \pi(z^n|z_0)a_0(z^n) = \Pi(z_0) + \widetilde{w}_0.$$

Using $\Pi(z_0) = -k^*(z_0) + \beta E_0\left[z_1 \left(k^*(z_{t+j})\right)^\alpha\right] - \overline{c} + \beta E_0\left[\Pi(z_1)\right]$ and (6) the previous equation delivers

$$(1 - \beta)E_0\left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + (1 + r)p_0' a_0 = -k^*(z_0) + \widetilde{w}_0 - \overline{c}$$

Using $\beta = 1/(1 + r)$ this can be written as

$$(1 - \beta)\left(E_0\left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + \frac{1}{\beta}p_0 a_0 + b_0\right)$$
$$+ \frac{1}{1 + r}b_0 + p_0' a_0 = -k^*(z_0) + \widetilde{w}_0 \qquad (49)$$

From substituting (47) into (48), multiplying the result with $\pi(z^n|z_0)$ and summing over all n, it follows that the terms in the first line of the previous equation are equal to

$$(1 - \beta)\left(E_0\left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + \frac{1}{\beta}p_0' a_0 + b_0\right) = \widetilde{c} - \overline{c}$$

where we also used (6) and $1 + r = 1/\beta$. Using this result to substitute the first line in (49) shows that (49) is just the flow budget equation for period zero. This proves that the portfolio giving rise to (48) in $t = 1$ for all $n = 1, ..., N$ satisfies the budget constraint of period $t = 0$. The results for $t \geq 1$ follow by induction. The result (15) follows from substituting (47) into (48) and noting that b_0 is not state contingent.

From equation (48) and the fact that $\Pi(z_t)$ is bounded, it follows that $\widetilde{w}_t$ is bounded so that the process for beginning-of-period wealth automatically satisfies the marginally binding NBLs. The multipliers ω_{t+1} in appendix A.1 are thus equal to zero for all t and all contingencies. Using $v_t(z^n) \equiv 0$, $\omega_{t+1} \equiv 0$, the fact that capital investment is given by (14) and that the Arrow security price is (6), the Euler conditions (26a) - (26c) all hold when consumption is given by (13). This completes the proof.

A.6 Proof of proposition 5

Since we start with a beginning-of-period wealth level at the marginally binding NBL, we necessarily have $\widetilde{c}_t = 0$ (otherwise one could afford an even lower initial wealth level and satisfy all constraints, which would be inconsistent with the definition of the marginally binding NBLs given in (11)). Indeed, the available beginning of period wealth $\widetilde{w}_t$ is just enough to insure that $\widetilde{w}_{t+1} \geq NBL(z_{t+1})$ for all possible future productivity states z_{t+1}. The optimal choices $a_t \in R^N, b_t, \widetilde{k}_t$ are thus given by the cost-minimizing choices satisfying $a_t \geq 0$ plus the marginally binding NBLs in $t+1$ for all possible future productivity states. Formally,

$$\min_{a_t, b_t, \widetilde{k}_t} \widetilde{k}_t + \frac{1}{1+r} b_t + \sum_{i=1}^{N} a_t(z^n) p_t(z^n)$$

$$s.t.$$

$$z^n \widetilde{k}_t^{\alpha} + b_t + (1-\lambda) a_t(z^n) - NBL(z^n) \geq 0 \text{ for } n = 1, ..., N$$

$$a_t(z^n) \geq 0 \text{ for } n = 1, ..., N$$

The optimal choices are thus equivalent to those defining the NBLs in Appendix A.3, see problem (35). As shown in that appendix, the optimal choices are given by (30)-(34) provided (12) holds.

A.7 Proof of proposition 6

Using the assumed policies, $\frac{1}{1+r} = \beta$, $p_t(z^n) = \frac{\pi(z^n|z_t)}{1+r}$, and the fact that the NBLs are not binding for sufficiently high wealth levels, the Euler equations (26a)-(26c) for $j = 0$ imply

$$u'(\widetilde{c}_t) = E_t u'(\widetilde{c}_{t+1}) \tag{50a}$$

$$\nu_t(z^n) = \beta \pi(z^n|z_t) \left(u'(\widetilde{c}_t) - u'(\widetilde{c}_{t+1}(z^n))(1-\lambda) \right) \quad \forall n \in N \tag{50b}$$

$$0 = -u'(\widetilde{c}_t) + \alpha \widetilde{k}_t^{\alpha-1} \beta E_t u'(\widetilde{c}_{t+1}) z_{t+1} \tag{50c}$$

We show below that the Euler equation errors for $j = 0$ converge to zero and that $v_t(z_t^n) \geq 0$ as the wealth $\widetilde{w}_t$ increases without bound. Under the assumed policies wealth evolves according to

$$\widetilde{w}_{t+1} = z_{t+1} k^*(z_t)^{\alpha} + b_t$$

$$= z_{t+1} k^*(z_t)^{\alpha} + \frac{1}{\beta} \left(\widetilde{w}_t - k^*(z_t) - (1-\beta)(\Pi(z_t) + \widetilde{w}_t) - \bar{c} \right)$$

$$= \widetilde{w}_t + z_{t+1} k^*(z_t)^{\alpha} - \frac{1}{\beta} k^*(z_t) - \frac{(1-\beta)}{\beta} \Pi(z_t) - \frac{1}{\beta} \bar{c}$$

$$= \widetilde{w}_t + \Pi(z_t) + z_{t+1} k^*(z_t)^{\alpha} - \frac{1}{\beta} k^*(z_t) - \frac{1}{\beta}(\Pi(z_t)) - \frac{1}{\beta} \bar{c}$$

$$= \widetilde{w}_t + \Pi(z_t) - E_t[\Pi(z_{t+1})] + (z_{t+1} - E_t z_{t+1}) k^*(z_t)^{\alpha}$$

Since the fluctuations in $z_t, k^*(z_t)$ and $\Pi(z_t)$ are all bounded, fluctuations in wealth are also bounded over any finite number of periods. Moreover, the fluctuations in wealth

are independent of the initial wealth level. As a result, fluctuations in consumption are also bounded and of a size that is not dependent on the wealth level under the proposed consumption policy.

We now show that for $j = 0$ and a sufficiently high wealth level the Euler equation residuals remains below ϵ. Using the assumed consumption policy and the result from the previous equation, we have

$$
\begin{aligned}
E_t[\widetilde{c}_{t+1}] &= (1 - \beta) E_t \left[(\Pi(z_{t+1}) + \widetilde{w}_{t+1}) \right] \\
&= (1 - \beta) \left(\widetilde{w}_t + \Pi(z_t) \right) \\
&= c_t
\end{aligned}
$$

i.e. consumption follows a random walk. Now consider equation (50a), which requires

$$
\begin{aligned}
u'(\widetilde{c}_t) &= E_t u'(\widetilde{c}_{t+1}) \\
&= \sum_{n=1}^{N} \pi(z^n | z_t) u'(\widetilde{c}_{t+1}(z^n))
\end{aligned}
\tag{51}
$$

From Taylor's theorem we have

$$
u'(\widetilde{c}_{t+1}(z^n)) = u'(\widetilde{c}_t) + u''(c_n)(\widetilde{c}_{t+1}(z^n) - \widetilde{c}_t)
$$

where c_n can be chosen from the bounded interval

$$
[\min\{\widetilde{c}_t, \min_n \widetilde{c}_{t+1}(z^n)\}, \max\{\widetilde{c}_t, \max_n \widetilde{c}_{t+1}(z^n)\}]
$$

whose width is independent of the wealth level $\widetilde{w}_t$ (as fluctuations in consumption do not depend on wealth as shown above). Moreover, under the assumed consumption policy, the lower bound of this interval - and thus also c_n increases without bound, as $\widetilde{w}_t$ increases without bound. Using the earlier result, (51) can be rewritten as

$$
0 = \sum_{n=1}^{N} \pi(z^n) \left(u''(c_n)(\widetilde{c}_{t+1}(z^n) - \widetilde{c}_t) \right)
$$

where the sum on the right-hand side of the equation denotes the Euler equation residual whenever it is not equal to zero. For the considered consumption policies, we have that $\widetilde{c}_{t+1}(z^n) - \widetilde{c}_t$ is bounded and invariant to wealth. Moreover, for sufficiently large wealth, c_n increases without bound, therefore $u''(c_n) \to 0$ under the maintained assumption about preferences. This implies that for any given $\epsilon > 0$ we can find a wealth level w^* so that the Euler equation residual falls below ϵ. Since the fluctuations in wealth are bounded over the finite horizon T and do not depend on the initial wealth level, we can find an initial wealth level $\bar{w}$ high enough such that over the next T periods wealth stays above w^*. The Euler equation errors then remain below ε over the next $T - 1$ periods, as claimed.

Similar arguments can be made to show that (50c) holds and that (50b) implies $v_t(z_t^n) \geq 0$ for a sufficiently large initial wealth level. We omit the proof here for the sake of brevity.

A.8 Default Costs Born by Lender

This appendix shows that if a consumption allocation is feasible in a setting in which default costs are borne by the borrower, then it is also feasible in a setting in which some or all of these costs are borne by the lender instead. For simplicity, we only consider the extreme alternative where all costs are born by the lender. Intermediate cases can be covered at the cost of some more cumbersome notation.

Consider a feasible choice $\{F_t \geq 0, D_t \geq 0, \Delta_t \in [0,1]^N, k_t \geq 0, c_t \geq 0\}_{t=0}^{\infty}$, i.e. a choice that satisfies the constraints of the government's problem (5), which assumes $\lambda^l = 0$ and $\lambda^b = \lambda$. Let variables with a hat denote the corresponding choices in a setting in which the lender bears all default costs, i.e., where $\lambda^l = \lambda$ and $\lambda^b = 0$. We show below that it is then feasible to choose the same real allocation, i.e., to choose $\widehat{k}_t = k_t$ and $\widehat{c}_t = c_t$, provided one selects appropriate values for $\widehat{F}_t$, $\widehat{D}_t$ and $\widehat{\Delta}_t$.

First, note that in a setting where foreign investors bear all settlement costs, the interest rate $\widehat{R}(z_t, \widehat{\Delta}_t)$ on domestic bonds satisfies

$$1 + r = \frac{1 - (1 + \lambda) \sum_{n=1}^{n} \widehat{\delta}_t^n \Pi(z^n | z_t)}{\frac{1}{1 + \widehat{R}(z_t, \widehat{\Delta}_t)}} \tag{52}$$

where the denominator on the right-hand side denotes the issuance price of the bond and the numerator the expected repayment net of the lender's settlement cost. The previous equation thus equates the expected returns of the domestic bonds with the expected return on the foreign bond.

Next, consider the following financial policies:[52]

$$\widehat{\Delta}_t = (1 - \lambda) \Delta_t \frac{D_t}{\widehat{D}_t} \tag{53a}$$

$$\widehat{D}_t = \frac{1 + \widetilde{R}(z_t, \widehat{\Delta}_t)}{\widetilde{R}(z_t, \widehat{\Delta}_t)} \frac{R(z_t, \Delta_t)}{1 + R(z_t, \Delta_t)} D_t \tag{53b}$$

$$\widehat{F}_t = F_t + \widehat{D}_t - D_t \tag{53c}$$

As we show below, in a setting in which settlement costs are borne by the lender, the financial policies $\{\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t\}_{t=0}^{\infty}$ give rise to the same state-contingent financial payoffs as generated by the policies $\{F_t, D_t, \Delta_t\}_{t=0}^{\infty}$ in a setting in which settlement cost are borne by the borrower. Therefore, as claimed, the former policies allow the implementation of the same real allocations.

Consider the financial flows generated by the policy component $(\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t)$. In period t, the financial inflows are given by

$$\frac{\widehat{D}_t}{1 + \widehat{R}(z_t, \widehat{\Delta}_t)} - \widehat{F}_t$$

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[52]Lengthy but straightforward calcucations, which are available upon request, show that these policies satisfy $\widehat{\Delta}_t \in [0,1]^N$, although they may imply $\widehat{F}_t < 0$, which requires the government also to issue safe bonds, i.e. bonds that promise full repayment in the explicit and implicit component of their contract.

Using the definitions (53), it is straightforward to show that these inflows are equal to

$$\frac{1}{1 + R(z_t, \Delta_t)} D_t - F_t$$

which are the inflows under the policy (F_t, D_t, Δ_t) in a setting where settlement costs are borne by the lender.

We show next that the financial flows in $t + 1$ are also identical under the two policies. The financial inflows generated by the policy choices $(\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t)$ in some future contingency $n \in \{1, ...N\}$ in period $t + 1$ are given by

$$-\widehat{D}_t(1 - \widehat{\delta}_t^n) + \widehat{F}_t$$

From the first and last equation in (53), we determine that these flows are equal to

$$-(1 - (1 - \lambda)\delta_t^n)D_t + F_t$$

which are the inflows generated by the policy (F_t, D_t, Δ_t) in a setting where settlement costs are borne by the lender.

Finally, since the policies $\{F_t, D_t, \Delta_t\}_{t=0}^{\infty}$ satisfy the marginally binding natural borrowing limits in the government's problem (5), it must generate bounded financial flows, with the same therefore also applying for the policies $\{\widehat{F}_t, \widehat{D}_t, \widehat{\Delta}_t\}_{t=0}^{\infty}$. These policies thus also satisfy the marginally binding natural borrowing limits, which completes the proof.

A.9 Estimation of Lender's Default Costs

Consider a non-contingent one period bond that in explicit legal terms promises to repay one unit and that has an associated implicit default profile $\Delta = (\delta^1, ..., \delta^n) \in [0, 1]^n$. A risk-neutral foreign lender, who bears proportional default costs λ^b in the event of default and can earn the gross return $1 + r$ on alternative safe investments, will price this bond according to equation (23). As explained below, the asset pricing equation (23) can be used to obtain an estimate for λ^l.

We start by defining the *ex post* return epr_t on a government bond

$$1 + epr_t = \frac{1 - \sum_{n=1}^{N} \delta^n \Pi(z^n | z_t)}{\frac{1}{1 + R(z_t, \Delta)}},$$

which is the bond return that accounts for losses due to non-repayment but not for potential default costs. *Ex post* returns can be measured from financial market data. Using the previous equation to substitute $\sum_{n=1}^{N} (1 - \delta^n) \cdot \pi(z^n | z_t)$ on the r.h.s. of equation (23) and applying the unconditional expectations operator[53], one obtains

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[53]The expectations operator integrates over the set of possible histories $z^t = \{z_t, z_{t-1}, ...\}$.

$$\lambda^l = \frac{E\left[epr_t - r\right]}{E\left[(1 + R(z_t, \Delta)) \sum_{n=1}^{N} \delta^n \Pi(z^n|z_t)\right]}. \tag{54}$$

Information about the average excess return, which shows up in the numerator of the previous equation, can be obtained from Klingen, Weder, and Zettelmeyer (2004), who consider 21 emerging market economies over the period 1970-2000. Using data from table 3 in Klingen, Weder, and Zettelmeyer (2004), the average excess return varies between -0.2% and +0.5% for publicly guaranteed debt, depending on the estimation method used.[54],[55] We use the average of the estimated values and set $E[epr_t - r] = 0.15\%$.

We now turn to the denominator on the r.h.s. of equation (54). Using a first order approximation we obtain

$$E\left[(1 + R(z_t, \Delta)) \sum_{n=1}^{N} \delta^n \Pi(z^n|z_t)\right] \approx E\left[1 + R(z_t, \Delta)\right] E\left[\sum_{n=1}^{N} \delta^n \Pi(z^n|z_t)\right], \tag{55}$$

where the last term equals (again to a first order approximation)

$$E\left[\sum_{n=1}^{N} \delta^n \Pi(z^n|z_t)\right] \approx \Pr(\delta > 0) E[\delta|\delta > 0].$$

Using data compiled by Cruces and Trebesch (2011), who kindly provided us with the required information, we observe for the 21 countries considered in Klingen et al. (2004) and for the period 1970-2000 a total of 58 default events, thus the average yearly default probability equals 8.9%. The average haircut conditional on a default was 25%; these figures therefore imply

$$E\left[\sum_{n=1}^{N} \delta^n \Pi(z^n|z_t)\right] \approx 2.22\%.$$

The average *ex ante* interest rate $R(z_t, \Delta)$ appearing in equation (55) can be computed by adding to the average *ex post* return of 8.8% reported in table 3 in Klingen, Weder, and Zettelmeyer (2004) for publicly guaranteed debt, the average loss due to default, which equals 2.22% to first order, such that $R(z_t, \Delta) \approx 11.02\%$. Combining these results to evaluate λ^l in equation (54) delivers our estimate for the default costs accruing to lenders reported in the main text.

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[54] As suggested in Klingen, Weder, and Zettelmeyer (2004), we use the return on a three-year US government debt instrument as the safe asset, since it approximately has the same maturity as the considered emerging market debt.

[55] The fact that *ex post* excess returns on risky sovereign debt are relatively small or sometimes even negative is confirmed by data provided in Eichengreen and Portes (1986) who compute *ex post* excess returns using interwar data. The negative *ex post* excess returns likely arise due to the presence of sampling uncertainty: the high volatility of the nominal exchange rate makes it difficult to estimate the mean *ex post* excess returns.

A.10 Numerical Solution Approach

We solve the recursive version of the Ramsey problem from section 2.2.3 using global solution methods, so as to account for the non-linear nature of the optimal policies. The state space S of the problem is given by

$$S = \left\{ z^1 \times \left[NBL(z^1), w_{max} \right], ..., z^N \times \left[NBL(z^N), w_{max} \right] \right\}$$

where $NBL(z^n)$ denotes the marginally binding natural borrowing limits and w_{max} is a suitably chosen and sufficiently large upper bound for the country's wealth level.

We want to describe equilibrium in terms of time-invariant policy functions that map the current state into current policies. Hence, we want to compute policies

$$\widetilde{f} : (z_t, w_t) \rightarrow (\{c_t, k_t, b_t, a_t\}),$$

where their values (approximately) satisfy the optimality conditions derived in A.1. We use a time iteration algorithm where equilibrium policy functions are approximated iteratively. In a time iteration procedure, tomorrow's policy (denoted by f^{next}) is taken as given and solves for the optimal policy f today, which in turn is used to update the guess for tomorrow's policy. Convergence is achieved once $||f - f^{next}|| < \epsilon$, where we set $\epsilon = 10^{-5}$. We then set $\widetilde{f} = f$. In each time iteration step we solve for optimal policies on a sufficient number of grid points distributed over the continuous part of the state space. Between grid points we use linear splines to interpolate tomorrow's policy. Following Garcia and Zangwill (1981), we can transform the complementarity conditions of our first order equilibrium conditions into equations. For more details on the time iteration procedure and how complementarity conditions are transformed into equations, see, for example, Brumm and Grill (2010). To come up with a starting guess for the consumption policy, we use the fact that at the NBLs optimal consumption equals the subsistence level. We therefore guess a convex, monotonically increasing function g which satisfies $g(z^i, NBL(z^i)) = \bar{c} \; \forall i$ and use a reasonable guess for $g(z^i, w_{max})$.